NOISE, CAPM AND THE SIZE AND VALUE EFFECTS

Robert Arnott[a] and Jason Hsu[b],[*]

We model a continuous time one factor economy where stock prices are noisy proxies of the informationally efficient stock values. The pricing error process is modeled as a mean-reverting process, which gives us a well-defined notion of over-pricing (positive pricing error) and under-pricing (negative pricing error) in the market. We show that in this economy, cap-weighting is a sub-optimal portfolio strategy. This is because, in a cap-weighting scheme, portfolio weights are driven by market prices; as such, more weights are allocated to over-valued stocks and less weight to under-valued stocks.

More importantly, we show that the CAPM would be rejected in this one factor economy with noise. Regressing portfolio returns against market clearing portfolio returns, non-cap-weighted portfolios exhibit significant alpha on average!

Additionally, a value tilted or size tilted portfolio is predicted to outperform (risk-adjusted). By construction, value and size are not risk factors in our one factor economy. However, in the cross-section, large cap stocks and high price-to-book stocks (growth stocks) tend to underperform. This is because higher capitalization stocks and higher price-to-books stocks are indeed more likely to be stocks with positive pricing errors.

We note that prices are explicitly inefficient in our economy. However, the inefficiency does not lead to arbitrage opportunities. We carefully show conditions which prevent arbitrage in our informationally inefficient economy.

The paper contributes to the anomalies literature by showing that mean-reversion in stock returns and the Fama–French size and value effects are driven by the same market defect—pricing noise! This suggests that models, such as disposition effect and information

---

[a]Chairman, Research Affiliates, LLC; Editor-in-Chief, Financial Analyst Journal.

[b]Director of Research and Investment Management, Research Affiliates, LLC; Visiting Professor in International Finance, University of California at Irvine.

[*]Corresponding author. E-mail: hsu@rallc.com

herding, which can generate stock price over-reaction and therefore mean-reversion in stock prices, can also explain the value and size puzzle.

1   Introduction

The size and value effects have spurred spirited debates since Banz (1981) and Reinganum (1981) documented that smaller capitalization stocks tend to outperform (risk-adjusted) and Stattman (1980) and Roseberg *et al.* (1985) documented that high book-to-market stocks also outperform (risk-adjusted). Similarly, other ratios such as earnings-to-price, documented by Basu (1980) and dividend yield, documented by Razeff (1984), Shiller (1984), Blume (1980) and Keim (1985), also predict future performance. Fama and French (1992) show that size and value, along with market beta, capture well the cross-sectional variation in stock returns and subsume the explanatory powers of other financial variables. However, there is little consensus on whether size and value are proxies for hidden risk factors or anomalies.[1]

Fama (1988) argues that market efficiency tests are always joint tests. Therefore, it is never certain whether we reject market efficiency or the assumed pricing model. The rejection of stock prices as random walks by Poterba and Summers (1988), Debondt and Thaler (1985, 1987) and Lo and MacKinlay (1988) can be seen as rejections of market efficiency or rejections of constant risk premium assumptions. If equity risk premium was time varying over the business cycle then financial variables that are correlated with the macro peaks and troughs would predict future returns and random walk assumption would not hold.

There are no easy solutions to this problem. Evidences suggesting inefficiencies in aggregate can almost always be explained away by carefully constructed rational models. However, models based on irrational investor behaviors have found limited success in explaining other observed anomalies other than the ones they were crafted to resolve. Additionally, there is always concern that the anomaly would cease to persist once discovered and therefore be irrelevant to investors interested in a normative model on investment strategies.

Since it is very difficult to distinguish between anomalies and hidden risk factor explanations, model simplicity and the ability to address multiple empirical curiosities are key in modeling. In the paper, we construct a one factor economy and assume a simple stock return process where market prices are noisy relative to fundamental prices (pricing inefficiency, and therefore investor irrationality, is explicitly assumed) and derive a variety of interesting portfolio results. We show first that price-weighted portfolios underperform non-price-weighted portfolio, risk-adjusted. This explains the puzzle documented by Arnott *et al.* (2005), Shimizu and Tamura (2005) and Hsu and Campollo (2006), which find that cap-weighted market indexes underperform indexes constructed from company financials. We also show that the covariance with the market clearing portfolio does not explain cross-section return variance in this one factor economy and that passive portfolios can show significant alphas in CAPM time series regressions. This is consistent with the literature on the testing of CAPM. We also show that individual stocks can exhibit short-term random walk but longer horizon negative autocorrelation. This is consistent with the findings of Shiller (1984), Summers (1986),

Stambough (1986), and Fama and French (1988). Additionally we show that market cap and ratios like book-to-market and earnings-to-market can predict next period returns, which is consistent with the vast empirical literature on size, value, earnings-to-price, and dividend yield. Finally, we show that size and value factors (constructed ala Fama and French (1992)) can explain the cross-sectional variance in returns and largely eliminate the time series regression alpha.

We believe that our model represents a parsimonious explanation for a number of anomalies in the empirical literature. Certainly, we do not explain everything, and specifically we remain agnostic on whether hidden risk factors or other behavioral drivers also partially contribute to the size and value related effects. However, we do believe that we offer a compelling model and contribute to the understanding of these empirical puzzles.

2 Framework

We model a one factor economy, where the risk premium for a stock depends singularly on its exposure to one unobserved source of aggregate risk (F). Furthermore, we assume that the mark-to-market prices, P_t (market prices), deviate from the informationally efficient stock values, V_t.[2] Specifically, $P_t = V_t + e_t$—that is, market prices are noisy proxies for the informationally efficient values, which are assumed unobservable. The idiosyncratic pricing errors (e_t) are assumed to mean-revert to zero at the speed ρ. Consequently, a stock, with a market price greater than its efficient value, is over-valued and deliver less than its risk-adjusted fair return and vice versa as e_t mean-reverts. However, since e_t is mean zero, on the average, this price inefficiency has no impact on expected stock returns. Additionally, since e_t are idiosyncratic, a broad based portfolio equally weighted would have almost no aggregate mispricing relative to the efficient valuation.

By assumption, the market is not *informationally efficient*, so alpha strategies exist; though there are no arbitrage opportunities.[3] It is therefore, tacitly assumed that investors are not aware of the alpha opportunity (or do not take advantage of it sufficiently) and thus allow such opportunity to persist. Both the pricing error process and the efficient stock value process are given exogenously. We do not model the preferences and institutional structures which lead to these exogenous valuation processes. However, we appeal to the stock return anomaly studies in empirical asset pricing literature and behavioral finance literature to support our assumption that prices are not always fully informationally efficient. De Bondt and Thaler (1985, 1987) and Shiller (1981), find patterns of investor overreaction. Poterba and Summers (1986, 1988) find evidence of stock price mean reversion. Lo and MacKinlay (1990) and Lakonishok *et al.* (1994) find ourperformance for contrarian strategies.

We show, in this economy, that buying and holding the *market clearing* portfolio lead to inferior portfolio performance. This is because the market clearing (market) portfolio is capitalization weighted; as such, unwarranted additional weights are given to over-valued stocks and taken from under-valued stocks, resulting in lower future returns. On the other hand, a non-cap-weighted portfolio with the similar factor exposure would earn a comparably higher return. In general, any price-weighted portfolio would exhibit positive correlation between portfolio weights and over-valuation (positive e_t), which leads to underperformance against its fair return.

Furthermore, we show that portfolio strategies which condition on size (market cap) and value (price-to-book or price-to-earnings) offer superior performances. Specifically, small cap and value biases in a portfolio improve risk-adjusted return over time. We show that size and value anomalies are driven by the same phenomenon—pricing

noise. In the cross-section, higher capitalization tends to be correlated with over-valuation, which leads to both large cap and high price-to-book underperformance.

In this economy, the market portfolio has a negative alpha when regressed against the macro factor F due to over-weighting the over-valued and under-weighting the under-valued. When we regress individual stocks against the market portfolio in a CAPM time series regression, we find significant non-zero intercept for a large fraction of the population (well in excess of what could be statistically reasonable). Consequently, market *beta* does not explain stock returns in the cross-section.

Our results suggest that the Fama–French value and size factors are likely not proxies for hidden risk factors but are anomalies. Size and value premia are not risk related but driven entirely by pricing noise in the equity market, which also drives stock price mean-reversion and contrarian profits.

The paper contributes to the anomalies literature in three ways. (1) It suggests that the value and size factors can arise empirically (even in a one factor economy) if the market portfolio is a poor proxy for the one hidden risk factor. (2) It shows that the value and size puzzle and the stock price mean-reversion are anomalies driven by the same market imperfection and can arise quite naturally when stock prices are noisy. (3) It suggests that behavioral and rational[4,5] models which can generate stock price overreactions resulting in contrarian strategy profits, can also explain the value and size effect.

3   True stock value dynamics for individual stocks

We assume an economy with one aggregate source of risk and a finite number of securities. However, it

will become obvious that many of the key results do not depend on the pricing model nor the one fact assumption. The true stock value is unobservable. The dynamics are described by

$$\frac{dV_i}{V_i} = \mu_i dt + \beta_i \sigma_F dW_F + \sigma_{\upsilon i} dW_{\upsilon i}, \quad (1)$$

where,

(1) μ_i is the drift term and is the instantaneous return for the true value process and is described by

$$\mu_i = r_f + \beta_i \lambda_F, \quad (2)$$

where r_f is the instantaneous risk free rate and λ_F is the risk premium for holding one unit of the factor risk exposure. Note that the risk premium formula is assumed. If the true stock price were observable and tradable, then (2) arises natural in equilibrium in the limit following the APT argument. We do not need this explicit relationship between factor exposure and expected return to drive most of our results. However, this relationship between factor loading and return is intuitively appealing and will be necessary for analyzing the cross-section return variance and time series analysis in a CAPM context.

(2) β_i is stock i's factor loading.

(3) dW_F is an increment to a standard Wiener process and represents the common factor to all stocks.

(4) $dW_{\upsilon i}$ is an increment to a standard Wiener process and represent idiosyncratic shocks to the true stock value. Additionally, we assume that $E[dW_{\upsilon i} dW_{\upsilon j}] = 0$ for $i \neq j$ and $E[dW_{\upsilon i} dW_F] = 0$.

Note that there is only one risk factor in this economy and risk premium can only be earned from holding exposure to this one factor risk.

4 Mark-to-market return dynamics for individual stocks

We further assume that the observed market price is a noisy proxy for the true stock value. The market price is defined by

$$P_i = V_i U_i, \tag{3}$$

where U_i is defined by

$$U_i = 1 + \tilde{U}_i, {}^{6} \tag{4}$$

where $\tilde{U}_i$ is a mean-reverting process defined by

$$d\tilde{U}_i = (1 + \tilde{U}_i)(-\rho_i \tilde{U}_i dt + \sigma_{\tilde{U}i} dW_{\tilde{U}i}), \tag{5}$$

where $0 \leq \rho_i < 1$ and $dW_{\tilde{U}i}$ is an increment to a standard Wiener process. Note that when $\tilde{U}_i > 0$, the market price is over-valued relative to the fair price. Additionally we assume that $E[dW_{\tilde{U}i}dW_{\tilde{U}j}] = 0$ for $i \neq j$, $E[dW_{\tilde{U}i}dW_{vj}] = 0$ for all i and j, and $E[dW_{\tilde{U}i}dW_F] = 0$.

The market price dynamics can then be written as

$$dP_i = V_i dU_i + U_i dV_i. \tag{6}$$

Substituting we have

$$dP_i = V_i U_i(-\rho_i \tilde{U}_i dt + \sigma_{\tilde{U}i} dW_{\tilde{U}i}) + U_i V_i(\mu_i dt$$
$$+ \beta_i \sigma_F dW_F + \sigma_{vi} dW_{vi}). \tag{7}$$

Rearranging, the mark-to-market return process is given by

$$dr_i = \frac{dP_i}{P_i} = (\mu_i - \rho_i \tilde{U}_i)dt$$
$$+ \beta_i \sigma_F dW_F + \sigma_{ri} dW_{ri}, \tag{8}$$

where

$$\sigma_{ri} dW_{ri} = \sigma_{\tilde{U}i} dW_{\tilde{U}i} + \sigma_{vi} dW_{vi}, \tag{9}$$

and where

$$\sigma_{ri} = \sqrt{\sigma_{\tilde{U}i}^2 + \sigma_{vi}^2}. \tag{10}$$

Note from Eq. (8), that the mean-reverting pricing error process does not have an impact on the equity premium; though the cumulative return does suffer from the increased volatility. From Eq. (8), the mark-to-market return process is mean-reverting, suggesting that observed stock returns are negatively autocorrelated. While empirical evidences support negative autocorrelation, the literature also concludes that the magnitude may be too small or the effect too unreliable to be profitably exploited given the volatility in stock returns. However, we concede that the mean-reversion in returns can be an uncomfortable prediction, especially in a partial equilibrium model. We appeal to Summers (1986) and argue that standard statistical tests cannot reject the random walk hypothesis even when the true process is strongly mean-reverting; as such investors would not take large positions to trade on any perceived mean-reversion in stock returns.

5 Mark-to-market return dynamics for portfolios

The return on a portfolio Ω defined by a vector of weights $\{\omega_1, \omega_2, \ldots, \omega_N\}$ can be written as

$$dr_\Omega = \sum_{i=1}^{N} \omega_i dr_i = (\mu_\Omega - \rho \tilde{U}_\Omega)dt + \beta_\Omega \sigma_F dF$$
$$+ \sigma_\Omega dW_\Omega, \tag{11}$$

where

$$\mu_\Omega = \sum_{i=1}^{N} \omega_i \mu_i = r_f + \beta_\Omega \lambda, \tag{12}$$

$$\rho \tilde{U}_\Omega = \sum_{i=1}^{N} \omega_i \rho_i \tilde{U}_i, \tag{13}$$

$$\beta_\Omega = \sum_{i=1}^{N} \omega_i \beta_i, \tag{14}$$

$$\sigma_\Omega dW_\Omega = \sum_{i=1}^{N} \omega_i \sigma_{ri} dW_{ri}, \tag{15}$$

where

$$\sigma_\Omega = \sqrt{\sum_{i=1}^{N} \omega_i^2 \sigma_{ri}^2}, \qquad (16)$$

and where in the limiting case $\sigma_\Omega dW_\Omega \rightarrow 0$ as $N \rightarrow \infty$.

To derive additional portfolio implications we need to make explicit the portfolio weighting scheme. In the following two sections, we consider the portfolio return dynamics for a cap-weighted portfolio and a non-cap-weighted portfolio.

5.1   Cap-weighted portfolios

For simplicity and without loss of generality, we assume each company issues only 1 share of stock (therefore market price and market cap are the same). The cap-weighted portfolio is defined by the following vector of weights

$$CW = \left\{ \frac{P_1}{P_\Sigma}, \frac{P_2}{P_\Sigma}, \cdots, \frac{P_N}{P_\Sigma} \right\}, \qquad (17)$$

where

$$P_\Sigma = \sum_{i=1}^{N} P_i, \qquad (18)$$

The return on the cap-weighted portfolio is then

$$dr_{CW} = (\mu_{CW} - \rho \tilde{U}_{CW})dt + \beta_{CW}\sigma_F dF$$
$$+ \sigma_{CW}dW_{CW}, \qquad (19)$$

where

$$\mu_{CW} = \sum_{i=1}^{N} \frac{P_i}{P_\Sigma}\mu_i = r_f + \beta_{CW}\lambda, \qquad (20)$$

$$\rho \tilde{U}_{CW} = \sum_{i=1}^{N} \frac{P_i}{P_\Sigma}\rho_i \tilde{U}_i$$

$$= \sum_{i=1}^{N} \frac{V_i}{P_\Sigma}\rho_i(1 + \tilde{U}_i)\tilde{U}_i, \qquad (21)$$

$$\beta_{CW} = \sum_{i=1}^{N} \frac{P_i}{P_\Sigma}\beta_i, \qquad (22)$$

$$\sigma_{CW}dW_{CW} = \sum_{i=1}^{N} \frac{P_i}{P_\Sigma}\sigma_{ri}dW_{ri}, \qquad (23)$$

and where $\sigma_{CW}dW_{CW} \rightarrow 0$ as $N \rightarrow \infty$.

Rewriting the drift term for the portfolio dynamics in (19), we have

$$\left(\mu_{CW} - \sum_{i=1}^{N} \frac{V_i}{P_\Sigma}\rho_i \tilde{U}_i^2 \right) - \sum_{i=1}^{N} \frac{V_i}{P_\Sigma}\rho_i \tilde{U}_i, \quad (24)$$

where $-\sum_{i=1}^{N} \frac{1}{P_\Sigma}\rho_i V_i \tilde{U}_i^2$ is strictly negative except when $\rho_i = 0$ for all i (when pricing errors are not mean-reverting but random walks). That is to say that cap-weighting leads to a drag in portfolio expected return.

While there are only a finite number of stocks (this is both realistic and necessary to prevent arbitrage in our economy), the exposition is more clear when we examine the limiting case expression. We will appeal to this format of analysis throughout the paper to improve intuition, though it is not necessary for the results.

In the limiting case, $\sum_{i=1}^{N} \frac{V_i}{P_\Sigma}\rho_i \tilde{U}_i \rightarrow 0$ as $N \rightarrow \infty$ and $\sum_{i=1}^{N} \frac{V_i}{P_\Sigma}\rho_i \tilde{U}_i^2 \rightarrow \delta_{CW}$.[7] Note δ_{CW} is monotone increasing in the average variance of the pricing noise in the stock cross-section. Equation (19) then reduces to

$$dr_{CW} = (\mu_{CW} - \delta_{CW})dt + \beta_{CW}\sigma_F dF. \quad (25)$$

And the holding period return is

$$E_t[r_{t,t+T}] = E_t e^{\int_t^{t+T} dr_{CW}}$$
$$= e^{(r_f + \beta_{CW}\lambda - \delta_{CW} - 0.5\beta_{CW}^2 \sigma_F^2)T}. \quad (26)$$

Equation (25) suggests that in a well diversified portfolio constructed from cap-weighting, the portfolio expected return is the cap-weighted expected

returns of the constituent stocks less a drag term δ_{CW}. This return drag occurs because portfolio weights are positively correlated with prices; stocks that are over-valued would receive added weights in the portfolio and stocks that are under-valued would receive less weights. The greater the mispricing in the market, the more severe is this problem and the larger the resulting drag (δ_{CW}) to the cap-weighted portfolio.

5.2   Market-value-indifferent portfolios

We now consider portfolio weights which do not depend on market capitalizations (or market prices). The weights could be as arbitrary as random weights or as simple as equal weights. We denote the vector of weights as

$$\mathrm{NC} = \{w_1, w_2, \ldots, w_N\}, \qquad (27)$$

The return on the non-cap-weighted portfolio is then

$$dr_{\mathrm{NC}} = (\mu_{\mathrm{NC}} - \rho\tilde{U}_{\mathrm{NC}})dt + \beta_{\mathrm{NC}}\sigma_F dF$$
$$+ \sigma_{\mathrm{NC}}dW_{\mathrm{NC}}, \qquad (28)$$

where

$$\mu_{\mathrm{NC}} = \sum_{i=1}^{N} w_i\mu_i = r_f + \beta_{\mathrm{NC}}\lambda, \qquad (29)$$

$$\rho\tilde{U}_{\mathrm{NC}} = \sum_{i=1}^{N} w_i\rho_i\tilde{U}_i, \qquad (30)$$

$$\beta_{\mathrm{NC}} = \sum_{i=1}^{N} w_i\beta_i, \qquad (31)$$

$$\sigma_{\mathrm{NC}}dW_{\mathrm{NC}} = \sum_{i=1}^{N} w_i\sigma_{ri}dW_{ri}. \qquad (32)$$

Note that the non-cap-weighted portfolio drift term is

$$\mu_{\mathrm{NC}} - \sum_{i=1}^{N} w_i\rho_i\tilde{U}_i. \qquad (33)$$

Comparing Eqs. (33) to (24), we find that a non-cap-weighted portfolio does not suffer a drag in expected return.

In the limit, $\sigma_{\mathrm{NC}}dW_{\mathrm{NC}} \to 0$ and $\rho\tilde{U}_{\mathrm{NC}} \to 0$ as $N \to \infty$.[8] Equation (28) then reduces to

$$dr_{\mathrm{NC}} = \mu_N dt + \beta_{\mathrm{NC}}\sigma_F dF. \qquad (34)$$

And the holding period return is

$$E_t[r_{t,t+T}] = E_t e^{\int_t^{t+T} dr_{\mathrm{NC}}}$$
$$= e^{(r_f + \beta_{\mathrm{NC}}\lambda - 0.5\beta_{\mathrm{NC}}^2\sigma_F^2)T}. \qquad (35)$$

Comparing the expected cumulative holding period return for a cap-weighted portfolio and a non-cap-weighted portfolio of the same factor exposure or same β (the limiting case shown in (26) and (35)), we find that the non-cap-weighted portfolio has a higher return. In fact, in the limit, there is arbitrage as indicated by (34) and (25). Therefore, it is important that in our economy, N is sufficiently different from infinity and/or that the factor loading β cannot be measured with perfect precision.

6   Using the cap-weighted "market" portfolio as a factor

In this section we express return dynamics for stocks and portfolios relative to the observed cap-weighted "market" portfolio instead of the unobserved factor F. This shift in measure leads naturally to the CAPM regression formula and predicts that in the stock cross-section, the average stock will show a CAPM alpha.

Rewriting Eq. (19), we have

$$\sigma_F dF = \frac{1}{\beta_{\mathrm{CW}}}dr_{\mathrm{CW}} - \frac{(\mu_{\mathrm{CW}} - \rho\tilde{U}_{\mathrm{CW}})}{\beta_{\mathrm{CW}}}dt$$
$$- \frac{\sigma_{\mathrm{CW}}}{\beta_{\mathrm{CW}}}dW_{\mathrm{CW}}. \qquad (36)$$

For individual stocks, substituting into (8), we have

$$dr_i = \left(\mu_i - \rho_i \tilde{U}_i - \frac{\beta_i}{\beta_{CW}} (\mu_{CW} - \rho \tilde{U}_{CW}) \right) dt$$
$$+ \frac{\beta_i}{\beta_{CW}} dr_{CW} - \frac{\beta_i}{\beta_{CW}} \sigma_{CW} dW_{CW}$$
$$+ \sigma_{ri} dW_{ri}. \tag{37}$$

Additionally, we define a new process, the excess market return process

$$dR_M = dr_{CW} - r_f dt, \tag{38}$$

and a new variable $\gamma_i = \frac{\beta_i}{\beta_{CW}}$.

Substituting into (37), we have

$$dr_i = (\mu_i - \rho_i \tilde{U}_i - \gamma_i(\mu_{CW} - r_f - \rho \tilde{U}_{CW}))dt$$
$$+ \gamma_i dR_M - \gamma_i \sigma_{CW} dW_{CW} + \sigma_{ri} dW_{ri}. \tag{39}$$

Recall Eq. (2), where $\mu_i = r_f + \beta_i \lambda_F$, we can rewrite (39) as

$$dr_i = (r_f - \rho_i \tilde{U}_i + \gamma_i \rho \tilde{U}_{CW})dt + \gamma_i dR_M$$
$$- \gamma_i \sigma_{CW} dW_{CW} + \sigma_{ri} dW_{ri}. \tag{40}$$

In the limiting case as $N \to \infty$, we have

$$dr_i = (r_f - \rho_i \tilde{U}_i + \gamma_i \delta_{CW})dt + \gamma_i dR_M + \sigma_{ri} dW_{ri}. \tag{41}$$

Note that the average stock is expected to show an "alpha" equal to $\gamma_i \delta_{CW}$ when its excess stock return is regressed against the excess market return.

For a non-cap-weighted portfolio, Eq. (28) can be expressed as

$$dr_{NC} = (r_f - \rho \tilde{U}_{NC} + \gamma_{NC} \rho \tilde{U}_{CW})dt$$
$$+ \gamma_{NC} dR_M - \gamma_{NC} \sigma_{CW} dW_{CW}$$
$$+ \sigma_{NC} dW_{NC}. \tag{42}$$

In the limiting case as $N \to \infty$, we have

$$dr_{NC} = \left(r_f + \gamma_{NC} \delta_{CW} \right) dt + \gamma_{NC} dR_M. \tag{43}$$

A non-cap-weighted portfolio is expected to show an "alpha" in a CAPM regression.

7 Value and size predict returns but are not risk factors

In this section we show that, in this economy, size and value exposure in a stock or portfolio can be used to predict future returns. Specifically, small size exposure and value exposure lead to superior stock or portfolio returns, adjusting for "market" beta. By assumption, we are in a one risk factor economy; size and value are not risk factors. The observed alpha in a CAPM regression is driven purely by the return drag in the cap-weighted market portfolio.

Recall from (40) that the individual stock return dynamics can be written as

$$dr_i = (r_f - \rho_i \tilde{U}_i + \gamma_i \rho \tilde{U}_{CW})dt + \gamma_i dR_M$$
$$- \gamma_i \sigma_{CW} dW_{CW} + \sigma_{ri} dW_{ri}. \tag{44}$$

Examining Eq. (44), we know that a larger stock would on average have a negative drift term in excess of the risk free r_f. It is straightforward to show that a larger stock, denoted by $p_i > \bar{p}$, where $\bar{p}$ denote the capitalization of the average company, will have a greater chance of receiving a positive pricing error $\tilde{U}$ in the last period and therefore be more likely to underperform going forward as the positive pricing error reverts to zero.

More formally, since $\tilde{U}_i$ is a mean zero random variable, $E[\tilde{U}_i | P_i > \bar{P}] > 0$ if the conditional probability $\Pr\{\tilde{U}_i > 0 | P_i > \bar{P}\} > \Pr\{\tilde{U}_i > 0\}$.

Using Bayes rule of conditional probability, we have:

$$\Pr\{\tilde{U}_i > 0 | P_i > \bar{P}\}$$
$$= \frac{\Pr\{P_i > \bar{P} | \tilde{U}_i > 0\} \cdot \Pr\{\tilde{U}_i > 0\}}{\Pr\{P_i > \bar{P}\}}. \tag{45}$$

It is clear that:

$$\Pr\{P_i > \bar{P} | \tilde{U}_i > 0\} > \Pr\{P_i > \bar{P}\}. \tag{46}$$

Substituting (46) into (45), we have

$$\Pr\{\tilde{U}_i > 0 | P_i > \bar{P}\}$$

$$= \frac{\Pr\{P_i > \bar{P} | \tilde{U}_i > 0\} \cdot \Pr\{\tilde{U}_i > 0\}}{\Pr\{P_i > \bar{P}\}}$$

$$> \Pr\{\tilde{U}_i > 0\}, \tag{47}$$

which completes the proof that $E[\tilde{U}_i | P_i > \bar{P}] > 0$. This in turn proves that size predicts next period return, $E[\int_t^{t+\Delta} dr_i | P_{i,t} > \bar{P}_t] < E[\int_t^{t+\Delta} dr_i]$.

Similarly, we show that, under some fairly general and reasonable assumptions on the book value process, a growth stock (as defined by above average price-to-book ratio or $\frac{P_i}{B_i} > \frac{\bar{P}}{B}$) would be more likely to have received a positive pricing error and therefore have a negative drift term in excess of the risk free r_f. We now show that

$$E\left[\tilde{U}_i \middle| \frac{P_i}{B_i} < \frac{\bar{P}}{B}\right] < 0 \quad \text{and} \quad E\left[\tilde{U}_i \middle| \frac{P_i}{B_i} > \frac{\bar{P}}{B}\right] > 0.$$

Again, we must show that $\Pr\left\{\tilde{U}_i > 0 \middle| \frac{P_i}{B_i} > \frac{\bar{P}}{B}\right\} > \Pr\{\tilde{U}_i > 0\}$ to prove that $E\left[\tilde{U}_i \middle| \frac{P_i}{B_i} > \frac{\bar{P}}{B}\right] > 0$.

First, Bayes rule gives:

$$\Pr\{\tilde{U}_i > 0 | P_i > \bar{P}\}$$

$$= \frac{\Pr\left\{\frac{P_i}{B_i} > \frac{\bar{P}}{B} \middle| \tilde{U}_i > 0\right\} \cdot \Pr\{\tilde{U}_i > 0\}}{\Pr\left\{\frac{P_i}{B_i} > \frac{\bar{P}}{B}\right\}}. \tag{48}$$

Now we need to show that

$$\Pr\left\{\frac{P_i}{B_i} > \frac{\bar{P}}{B} \middle| \tilde{U}_i > 0\right\} > \Pr\left\{\frac{P_i}{B_i} > \frac{\bar{P}}{B}\right\}. \tag{49}$$

A sufficient condition for this inequality to hold is that the book value process B is not influenced by market mispricing $\tilde{U}_i$ as strongly as the price process P_i.[9] More specifically, as long as the process for $\frac{P_i}{B_i}$ has a drift term that is negative in $\tilde{U}_i$, the inequality bears true.

Hence, if the book values of companies are not (as) subjected to the effects of mispricings in stock prices, then $E\left[\tilde{U}_i \middle| \frac{P_i}{B_i} > \frac{\bar{P}}{B}\right] > 0$, which indicates that price-to-book ratio can predict next period return, $E\left[\int_t^{t+\Delta} dr_i \middle| \frac{P_{i,t}}{B_{i,t}} > \frac{\bar{P}_t}{B_t}\right] < E[\int_t^{t+\Delta} dr_i]$.

The inequality in Eq. (49) can be extended to include more than just price-to-book ratio but also price-to-dividend and price-to-earnings ratios. This further explains the empirical observations that low yielding stocks and high P/E stocks tend to underperform.

Since conditional expectation is linearly additive, it is again straight forward to show that any portfolio which has smaller weighted average cap than the "market" portfolio, would have a positive excess drift and would show a positive CAPM alpha in a time series regression. Similarly, any portfolio which has a lower price-to-book ratio (lower P/E or higher yield) than the "market" portfolio, would have a positive excess drift and show a positive CAPM alpha.

8 Conclusion

In this paper, we show that the Fama–French value and size factors can be explained quite simply if we are willing to entertain informational inefficiency in stock prices. A simple one factor economy with price noise, where pricing errors are mean-reverting, can generate the Fama–French return anomalies as well as mean-reversion in stock returns. Given the strong support in the empirical and the behavioral literature that point to excess price volatility (price overshooting) and contrarian profits, we find our explanation of the Fama–French size and value anomalies more attractive than explanations based on rational models with hidden risk factors.

The model is very parsimonious and is able to simultaneously explain stock price mean-reversion and

the size and value effects and is able to offer reasonable explanation for the empirical findings from the CAPM literature. We believe that the paper contributes significantly to the anomalies literature in three ways. (1) It suggests that the value and size factors can arise empirically (even in a one factor economy) if the market portfolio is a poor proxy for the one hidden risk factor. (2) It shows that the value and size puzzle and the stock price mean-reversion are anomalies driven by the same market imperfection and can arise quite naturally when stock prices are noisy. (3) It suggests that behavioral and rational models which can generate stock price overreactions resulting in contrarian strategy profits, can also explain the value and size effect.

Notes

[1] See Fama (1991) for a comprehensive review on anomalies and market efficiency.

[2] For our purpose, it is not necessary to define how the market arrives at the informationally efficient value V_t. However, it may still be convenient to think of the classic valuation equation where $V_t = E_t\left[\sum_{s=t}^{\infty} e^{-rs} Div_s \right]$.

[3] In classic efficient markets, the future prospects of an investment tacitly rise and fall with share price, so that the IRR of an investment will not be advantaged by a drop in price or disadvantaged by an increase. Our assumptions stand in stark contrast—when price rises there is an appreciable probability that the increase is driven partly by overreaction to positive news, therefore, the subsequent IRR can be expected to fall. Empirically, this results in stock price mean-reversion, which is a violation of the weak-form market efficiency condition.

[4] See papers on disposition effect by Sherfin and Statman (1985), Grinblatt and Han (2002) and Goetzmann and Massa (2005) and on mental accounting bias by Kahneman and Tversky (1979) and Barberis and Huang (2001).

[5] See papers on rational and irrational information herding by Khanna and Sonti (2000), Hautière (2002), Chan et al. (2005), Lakonishok (2002), and Maug and Naik (1996).

[6] This substitution is for notational ease and simplicity of exposition, but does not otherwise further any useful intuition.

[7] The limit converges to zero if $\frac{V_N}{P_\Sigma}$ does not fall faster than $1/N$ as N goes to ∞.

[8] Again, for the limit to converge to zero, we require w_i to fall slower than $1/N$ as N goes to ∞.

[9] A reasonable process for book value might be one which depends on the unobserved efficient value V as well other variables that do not depend on the pricing error $\tilde{U}_i$: $\frac{dB_i}{B_i} = \kappa_i \frac{dP_i}{P_i} + \eta_i dt + \sigma_{Bi} dW_{Bi}$, where $\kappa_i < 1$.

References

Arnott, R.D., Hsu, J.C., and Moore, P. (2005). "Fundamental Indexation." *Financial Analysts Journal* **61**(2).

Arnott, R.D. (2005a). "Editor's Corner: What Cost Noise?" *Financial Analysts Journal* **61**(2).

Arnott, R.D. (2005b). "Editor's Corner: Disentangling Size and Value." *Financial Analysts Journal* **61**(5).

Berk, J.B. (1997). "Does Size Really Matter?" *Financial Analysts Journal* **52**(5).

Berk, J.B. (1995). "A Critique of Size Related Anomalies." *Review of Financial Studies* (8) 275–286.

De Bondt, W.F.M. and Thaler, R. (1985). "Does the Stock Market Overreact?" *The Journal of Finance* **40**(3), 793–805.

De Bondt, W.F.M. and Thaler, R. (1987). "Further Evidence on Investor Overreaction and Stock Market Seasonality." *The Journal of Finance* **42**(3), 557–581.

Fama, E.F. and French, K.R. (1988). "Permanent and Temporary Components of Stock Prices." *Journal of Political Economy* **96**(2), 246–273.

Hsu, J.C. (2006). "Cap-Weighted Portfolios Are Sub-optimal Portfolios." *Journal of Investment Management*, 3ʳᵈ Quarter.

Hsu, J.C. and Campollo, C. (2006). "An Examination of Fundamental Indexation." *Journal of Indexes* Jan/Feb.

Jegadeesh, N. (1990). "Evidence of Predictable Behavior of Security Returns." *The Journal of Finance* **45**(3), 881–898.

Jegadeesh, N. and Titman, S. (1993). "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency." *The Journal of Finance* **48**(1), 65–91.

Kahneman, D. and Tversky, A. (1982). "Intuitive Prediction: Biases and Corrective Procedures." In Kahneman, D. Slovic, P. and Tversky, A. (eds.), *Judgment Under Uncertainty: Heuristics and Biases*, London: Cambridge University Press.

Kim, M.J., Nelson, C.R., and Startz, R. "Mean Reversion in Stock Prices? A Reappraisal of the Empirical Evidence." *Review of Economic Studies* **58**, 515–528.

Lakonishok, J., Shleifer, A., and Vishny, R. (1994). "Contrarian Investment, Extrapolation, and Risk." *The Journal of Finance* **49**(5), 1541–1578.

La Porta, R., Lakonishok, J., Shleifer, A., and Vishny, R. (1997). "Good News for Value Stocks: Further Evidence on Market Efficiency." *The Journal of Finance* **52**(2) 859–874.

Lo, A.W. and Craig MacKinlay, A. (1988). "Stock Market Prices do not Follow Random Walks: Evidence from a Simple Specification Test." *The Review of Financial Studies* **1**(1), 41–66.

Lo, A.W. and Craig MacKinlay, A. (1990). "When are Contrarian Profits Due to Stock Market Overreaction?" *The Review of Financial Studies* **3**(2), 175–205.

Markowitz, H.M. (2005). "Market Efficiency: A Theoretical Distinction, and So What?" *Financial Analysts Journal* **61**(5), 17–30.

Poterba, J.M. and Summers, L.H. (1986). "The Persistence of Volatility and Stock Market Fluctuations." *The American Economic Review* **76**(5), 1142–1151.

Poterba, J.M. and Summers, L.H. (1988). "Mean Reversion in Stock Prices: Evidence and Implications." *Journal of Financial Economics* **22**, 27–59.

Shiller, R. (1981). "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?" *American Economic Review* **71**, 421–436.

Shimizu, Y. and Tamura, H. (2005). "Fundamental Indices, Do they Outperform Market-Cap Weighted Indices on a Global Basis?" *Security Analysts Journal* **43**(10).

Summers, L. (1986). "Does the Stock Market Rationally Reflect Fundamental Values?" *The Journal of Finance* **41**(3), 591–601.

Treynor, J. (2005). "Why Market-Valuation-Indifferent Indexing Works." *Financial Analysts Journal* **61**(5).