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A Total Differential Approach to Equity Duration

Most of the risk associated with fixed income price movements is accounted for by their duration—that is, their sensitivity to changes in the discount rate. Thus, for bonds, duration and interest rate sensitivity are virtually synonymous. For equities, however, duration is only one of several factors describing risk.

A major source of confusion in evaluating equity duration is the definition of "duration" itself. Measured as a function of the sensitivity of stock price to the discount rate—ignoring all links between the discount rate and the growth rate—the traditional dividend discount model (DDM) duration is long—20 years or more. But this measure of duration fails to take into account the offsetting effects of inflation-induced rate increases on corporate profits.

A measure of the total sensitivity of stock prices to interest rate movements recognizes that the two components of nominal interest rates—inflation and the real rate—affect both the equity discount rate and the equity earnings growth rate, but not necessarily in the same direction. The stock market as a whole is less sensitive to changes in inflation expectations than to changes in real rates, because most companies can raise prices, hence nominal growth rates, in times of inflation. Thus a measure of stock price total sensitivity to interest rates will generally be substantially shorter than the duration measure derived from the traditional DDM.

THE DURATION of an equity portfolio—its sensitivity to the discount rate—has a meaningful impact on its relative performance. Perhaps more importantly, the duration of a portfolio has a profound effect on the match (or mismatch) between the assets in a portfolio and the interest rate sensitivity of the liabilities covered by that portfolio. Financial Accounting Standards Board Statement No. 87 will increase the importance of this correlation between assets and liabilities for plan sponsors. Any mismatch between assets and liabilities will affect the bottom line by increasing the volatility in earnings and pension surplus. There is thus a need for a more precise measure of the contribution of equities to the duration of a total portfolio.

For fixed income securities, duration accounts for the majority of the risk associated with price movements. Thus duration and interest rate sensitivity are virtually identical properties in the context of bond price behavior. For equities, duration is only one of several important factors that describe risk. An understanding of equity duration is nevertheless necessary for managing assets effectively in a liability context.

Considerable confusion has arisen regarding the "proper" measure for equity duration. The early work on equity duration was derived from valuation techniques based on some form of the dividend discount model (DDM). Equity duration was taken as the elasticity of a stock's theoretical DDM value with respect to changes in the discount rate, or "internal rate of return."

The early DDM duration calculations typically

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led to values ranging from 20 to 50 years, with growth companies exhibiting the longest duration values. An alternative form of analysis using straightforward regression techniques has been used to estimate empirically actual stock price sensitivity to interest rate changes. This has led to "empirical duration" values that range between two and six years—significantly less than the duration estimates derived from the DDM approach.[1]

This apparent "paradox" has important implications for the DDM model, both in terms of its theoretical underpinnings and its practical applications. This article differentiates between a stock's duration, using the DDM discount rate, and its interest rate sensitivity. We begin with the basic valuation model and then show how certain relationships can help reconcile the concepts of stock duration and interest rate sensitivity.

Early Development of Duration

Duration measures the time horizon of an asset, based on the present-value-weighted average time to receipt of income or principal. As a direct result of this formulation, Hicks and Macaulay, in the late 1930s, demonstrated that duration is the elasticity of the value of a capital asset with respect to changes in the discount factor.[2]

Much of the original duration work focused on fixed income instruments. Strategies such as dedication and bond portfolio immunization were based explicitly on the use of duration to control the risk exposure of a bond portfolio.[3] It was discovered that the structure of a bond portfolio could be adjusted so that the duration of the assets precisely matched that of the contractual liabilities, thereby leading to a nearly risk-free fit between a bond portfolio and contractual liabilities.

This structure defines interest rate sensitivity only for bonds. Enlarging the asset framework by introducing equities into the portfolio alters the potential for immunization. More generally, the way in which we view equity duration affects the degree to which we can model total portfolio duration and, thereby, control interest rate risk.

1. Footnotes appear at end of article.

Stock Duration: The DDM Formulation

Stock valuation, like bond valuation, provides a framework for assessing duration. Most approaches rely on a valuation equation such as the following:

$$P = \sum_{t=1}^{\infty} \frac{D_t}{(1+k)^t}, \qquad (1)$$

where

P = the theoretical value of the stock,
D_t = the dividend at end of period t and
k = the discount rate.

This generalized valuation formula, which uses the dividend discount model, can be used to derive the duration of a dividend stream through iterative calculations. However, the mathematics of this formulation is complex.

The derivation of duration and calculation of sources of duration are much easier if we shift to a Gordon-Shapiro formulation for the dividend discount model. This simplification assumes that future dividends are determined by a constant growth rate. Equation (1) then becomes:

$$P = \sum_{t=1}^{\infty} \frac{D_0(1+g)^t}{(1+k)^t}, \qquad (2)$$

where g is the dividend growth rate.

The Gordon-Shapiro formulation for the DDM simplifies the derivation of and implications for equity duration, without straying too far afield from the more generalized DDM structure. Equation (2) can be modified in a number of ways to model the elasticity of value with respect to changes in the discount rate.[4] Equation (2) reduces to the well known growth formula:

$$P = \frac{D_0(1+g)}{k-g}. \qquad (3)$$

DDM duration, D_{DDM}, is evaluated by taking the derivative of the natural logarithm of P with respect to the discount rate. This results in:

$$D_{DDM} = -\frac{\partial \ln P}{\partial k} = \frac{1}{k-g}. \qquad (4)$$

A stock with a long-term growth forecast of 10 per cent and a discount rate of 14 per cent would in this case have a DDM duration of 1.0/0.04, or 25 years.

Misuse of DDM Duration

This formulation of DDM duration is overly simplistic because it lacks the dynamic elements of cause and effect. It assumes that the estimate of future growth in dividends (or earnings), g, is unrelated to changes in the discount rate, k.

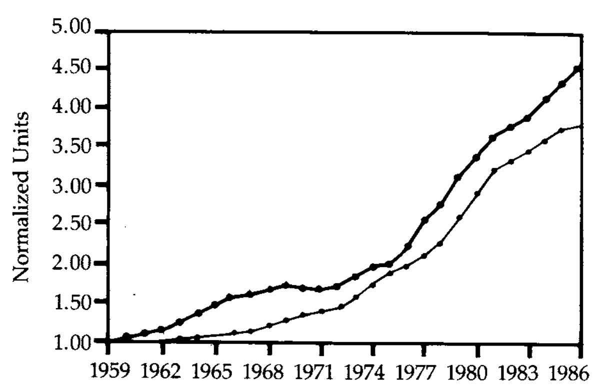
In reality, factors affecting the dividend growth rate will also affect the discount rate. A change in inflation, for example, may cause k to rise through the transmission mechanism of equilibrium interest rates; that is, as inflation changes, so do interest rates and capital market rates in general. But dividend growth may also respond to inflation, and this would tend to dampen the duration effect. In this case, Equation (4) would overstate true duration.

The total interest rate sensitivity of an asset relates to the impact of interest rate changes on the growth of future earnings, as well as their impact on the discount rate. Moreover, changes in the discount rate come not only from periodic shifts in interest rates, but also from changes in the overall equity market risk premium.

The duration estimates produced from Equation (4) assume that a stock's future dividend growth rate is constant. In the long run, with substantial smoothing, this may be a reasonable assumption. Over shorter intervals, however, constant growth is a poor assumption. Figure A provides a graphic representation of the 27-year history of growth in the S&P 500 dividend, as well as the history of inflation as measured by the Consumer Price Index (CPI).

It is apparent that dividends for the broad market do not grow at a constant rate, year to year. The long-term average growth in dividends is approximately 5.75 per cent, but there is a great deal of variability of growth. Over short and intermediate investment horizons, stock prices react to the expectation of variable future growth, rather than constant growth. The propensity for the growth rate to fluctuate implies that Equation (4) is not a realistic reflection of stock price response to interest rate change.

Figure A Dividend Growth and Inflation



Total Interest Rate Sensitivity

We approached equity duration from a different perspective from conventional DDM duration calculations. Our approach emphasizes the importance of covariance between changes in stock prices and changes in interest rates.[5] An important aspect of this work is the attempt to capture the statistical significance of a most important variable—actual price change. The analysis of total portfolio duration improves our ability to assess the link between stock market returns and bond market returns in targeting pension surplus.

A Model

A comprehensive model of equity duration requires a framework that encompasses the dynamics of both earnings growth and the equity market risk premium. Our initial assumption is that fluctuating real interest rates and the inflation component of interest rates are the underlying variables that relate changes in the interest rate and the equity risk premium to equity duration.

It is impossible to model all the possible factors that may affect discount rates, dividend growth rates and equity risk premiums.[6] We can, however, clearly identify inflation and real interest rates. These two factors alone can have a profound impact on equity duration.

To simplify the calculations, we assume that all rates are continuously compounded and all cash flows are received continuously (see the appendix). In this case, the links between the discount rate, k, the nominal interest rate, i, inflation, I, and the growth rate, g, can be expressed as:

$$k = i + h(I, r, \ldots), \tag{5}$$

$$i = r + I, \tag{6}$$

$$g = g_0 + \gamma r + \lambda I, \tag{7}$$

where

i = the nominal interest rate,
h = the equity market risk premium,
r = a real component of nominal rates,
I = an inflation component of nominal rates,
g_0 = a constant-growth parameter,
γ = the growth rate sensitivity to real interest rates and
λ = an inflation flow-through parameter.

The discount rate for equities, k, comprises a real return, r, an inflation rate, I, and an equity market risk premium, h. The first two factors relate directly to nominal yields in the bond market, and the third factor is the incremental discount rate in the equity market.

The growth function, g, is a modification of the constant-growth model. Variability in growth has two components. The parameter λ captures the inflation flow-through, which is the effect of inflation on the growth in corporate profits. The parameter γ measures the sensitivity of corporate profit growth rates to changes in real interest rates.

We have made the assumption that interest rates are driven by inflation and real interest rates. In addition, we have assumed that the equity discount rate and earnings growth rate are also influenced by these same two factors. It is now necessary to determine the total impact of changes in both the real rate of interest and the rate of inflation on the theoretical price. This is accomplished by computing the total differential of the price function.

Formally, the total differential for stock price is:

$$\frac{dP}{P} = -D_{DDM}\left(1 - \gamma + \frac{\partial h}{\partial r}\right) dr - D_{DDM}\left(1 - \lambda + \frac{\partial h}{\partial I}\right) dI. \qquad (8)$$

This formula may look a little forbidding! Fortunately, there is considerable intuition behind the formulation. (The appendix provides the mathematical details of how to determine the total differential of the price function.)

First, Equation (8) states that price sensitivity is directly related to the DDM model, in that the higher the DDM duration, the higher the sensitivity of the stock to interest rate change. Second, an investor should think of stock duration not only as it relates to *nominal* changes in interest rates, but also as it relates to *real* changes in interest rates.

The formula has two terms. The right-hand term says that, irrespective of a stock's DDM duration, companies with high levels of inflation flow-through (say, λ approaching one), may have very low interest rate sensitivities. The left-hand term says that, notwithstanding a stock's DDM duration, companies that are adversely affected by increases in real interest rates, because of financial or business-related reasons, will have accentuated interest rate sensitivity.

The Real Rate vs. Inflation

The dichotomy between the sensitivity of stock prices to inflation and their sensitivity to the real interest rate is best illustrated by an example. Suppose that $(k - g)$ is 4 per cent, so the D_{DDM} is 25 years. Assume that interest rates rise by 100 basis points, solely in response to investor expectations of rising inflation. In addition, assume that the equity risk premium does not change with changing inflation expectations ($\frac{\partial h}{\partial I} = 0$).

For the S&P 500, λ is, empirically, about 0.80.[7] In other words, 80 per cent of any change in inflation rates tends to "flow through" equities in the form of earnings growth. A 1 per cent inflation-induced rise in interest rates would thus cause the price of the S&P 500 to change as follows:

$$\frac{\Delta P}{P} \approx -D_{DDM}(1 - \lambda)\, \Delta I,$$
$$= -25(1 - 0.8)(1\%),$$
$$= -5\%.$$

Suppose, alternatively, that nominal rates of interest rise by 100 basis points as a result of a 100-basis-point increase in real interest rates. This produces very different results. Higher real rates may increase the cost of doing business. Furthermore, a rise in real interest rates should, all else equal, increase savings relative to consumption, thereby making it difficult for firms to recover the higher costs through product pricing. In such a scenario, the sensitivity of earnings growth to a change in real interest rates may be negative, so that $(1 - \gamma)$ would exceed unity.[8]

Suppose that a 100-basis-point increase in real interest rates reduces dividend growth rates by

20 basis points. (And assume that $\partial h/\partial r = 0$.) This would mean that a 100-basis-point increase in real interest rates would result in a drop in stock prices of:

$$\frac{\Delta P}{P} \approx -D_{DDM}(1 - \gamma)\,\Delta r$$

$$= -25(1 + 0.2)(1\%),$$

$$= -30\%.$$

Figures B and C illustrate the effects of λ and γ. We can see that the stock market is much less sensitive to changes in inflation expectations than to changes in the real rate of interest. This is because most companies can raise prices and, consequently, nominal growth rates of dividends in time of inflation.

Firms with low flow-throughs, such as electric utilities, exhibit large price swings when interest rates change. By contrast, firms that are perfectly indexed to inflation ($\lambda = 1$) exhibit little, if any, price sensitivity to changes in inflation expectations. These companies could, however, demonstrate great sensitivity to interest rate changes resulting from changes in the real rate of interest.

Dynamics of the Risk Premium

The equity market risk premium varies over time. The risk premium is typically measured by solving the dividend discount model for the discount rate, k.[9] Risk premiums change with shifts in investors' perceptions of risk and their tolerance of it. As stock prices rise or fall, the resulting expected return, when measured against other available returns such as those for cash or bonds, reflects changes in the risk premium.

Figure B Price Sensitivity vs. DDM Duration for Different Flow-Throughs

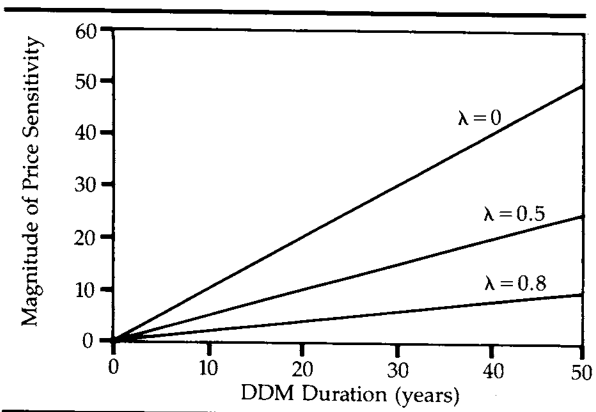
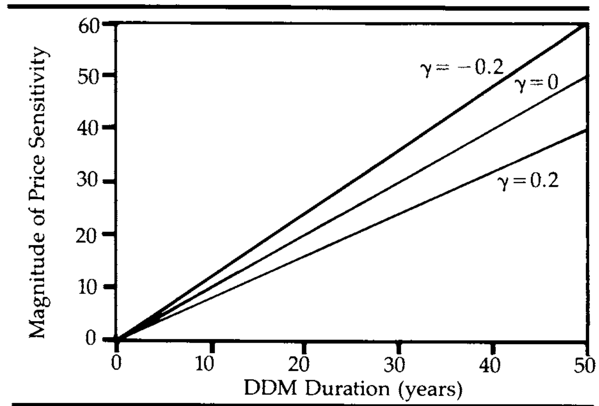


Figure C Price Sensitivity vs. DDM Duration for Different Real Rate Sensitivities



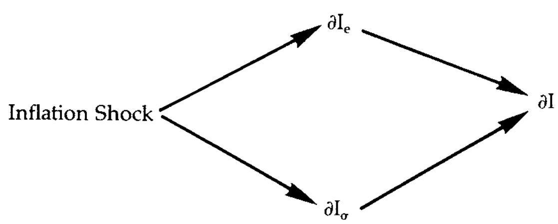
The risk premium has an important role in our duration model. DDM duration assumes that a change in the discount rate, k, is precipitated by either a change in interest rates or a change in the risk premium. The traditional DDM approach does not address the potential interaction between changing interest rate levels and changing equity risk premiums.

In Equation (8), ∂h represents the dynamic element of the risk premium. If the risk premium rises with an increase in inflation or real interest rates, then ∂h might be considered "duration-augmenting." If the risk premium falls as inflation or real rates rise, then ∂h might be considered "duration-dampening."

An analysis of the sources of returns leads us to focus on the equity risk premium and its response to changing conditions. Even if we equate risk premium changes with inflation changes, we are confronted with some surprising subtleties in the nature of ∂h.

The heart of the issue lies in the manner in which inflation shifts occur and the manner in which investors react to inflation. Consider a "burst" in inflation. On the one hand, a burst of inflation may be disruptive to investor expectations. This will undoubtedly increase economic uncertainty and, consequently, should cause the risk premium to rise. On the other hand, a burst in inflation may enhance investors' appetite for inflation protection. Because the income stream of stocks offers significant inflation protection, an inflation shock may actually reduce the risk premium for stocks.

Figure D The Nature of the Inflation Shock



It thus seems necessary to differentiate between two aspects of inflation. Consider a two-factor mechanism in which an external shock affects ∂I in the manner illustrated in Figure D. The first factor, ∂I_e, represents the interest rate change resulting from a change in perception about expected future inflation. The second factor, ∂I_σ, is unexpected inflation, or variability around the level of expected inflation. The impact of higher or lower inflation can be attributed jointly to changes in expected inflation and changes in uncertainty about future inflation.

Conclusions

One of the sources of confusion in evaluating equity duration stems from the definition of the word "duration" itself. If we measure duration as a function of the sensitivity of stock price to the discount rate, k, and ignore all links between the discount rate and the growth rate or other shared factors, the result is the traditional DDM duration—typically 20 years. If we measure the sensitivity of stock prices to interest rate movements, we get a very different result. The disparity is not a result of one formulation being incorrect, but stems from the use of different definitions for duration. Our analysis resolves the apparent paradox by demonstrating the sensitivity of stock prices to a variety of factors.

We have assumed that inflation and the real rate of interest are the media linking interest rate change with discount rate change, hence with total equity duration. Whereas DDM duration may be 20 years or more, corporate profits will be largely hedged against rises in inflation. As inflation rises, pressure on equity valuations through the discount rate will be offset to some extent by increases in the expected growth of nominal profitability. Thus the sensitivity of equity prices to inflation movements will tend to be far lower than the DDM duration.

The sensitivity of stock prices to shifts in real interest rates can, theoretically, be quite significant. Because real interest rates must be defined as long-term interest rates less the expected long-term rate of inflation, the level of real interest rates is impossible to measure accurately. We can, however, demonstrate that movements in real interest rates, so defined, have a profound effect on equity valuation—of a magnitude consistent with traditional DDM duration.

In the absence of inflation flow-through or any link between the equity risk premium and inflation, interest rate sensitivity and stock duration would be synonymous and very long. In reality, flow-through is a positive number. In the long term, it can approach 1.0 for the market as a whole. This inflation flow-through largely explains why the empirical duration of equities, as measured by interest rates, is so much shorter than the calculated DDM duration, as measured by the discount rate. This also suggests an important subtlety in the difference between inflation sensitivity and real interest rate sensitivity: Although interest rate changes that stem from shifts in inflation have only a modest effect on the stock market as a whole, changes in real interest rates can have a much more profound impact on equity pricing. ■

Footnotes

1. See M. L. Leibowitz, "Total Portfolio Duration: A New Perspective on Asset Allocation," *Financial Analysts Journal*, September/October 1986.
2. J. R. Hicks, *Value and Capital* (Cambridge: Oxford University Press, 1939) and F. R. Macaulay, *Some Theoretical Problems Suggested by the Movement of Interest Rates, Bond Yields, and Stock Prices Since 1856* (New York: National Bureau of Economic Research, 1938).
3. See L. Fisher and R. Weil, "Coping with the Risk of Interest Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies," *Journal of Business*, October 1972. See also M. Hopewell and G. Kaufman, "Bond Price Volatility and Term to Maturity: A Generalized Perspective," *American Economic Review*, September 1973.
4. See E. H. Sorensen and S. B. Kreichman, "Valuation Factors: Introducing the E-MODEL" (Salomon Brothers Inc, May 12, 1987).
5. See Leibowitz, "Total Portfolio Duration," *op. cit.*
6. See T. Estep, N. Hanson and C. Johnson, "Sources of Value and Risk in Common Stocks," *Journal of Portfolio Management*, Summer 1983.

7. We have estimated the flow-through of inflation for various industries. The range of estimates is wide, but the average across all industries from 1980 to 1987 was estimated to be 0.8. In addition, the estimates depend upon how the lead-lag structure is modeled.
8. Such a scenario is consistent with a real rate resulting from an unanticipated change in monetary policy. If, however, the rise in real rates is due to other exogenous forces, such as real economic expansion, then earnings growth sensitivity may not be negative.
9. See E. H. Sorensen and R. D. Arnott, "The Equity Risk Premium and Stock Market Performance," *Journal of Portfolio Management*, Summer 1988.

Appendix

Macaulay's duration is defined to be the present-value-weighted average time to receipt of a payment. Using our convention of continuous compounding and cash flows, duration is given by:

$$D = \frac{1}{P}\int_0^T td(t)e^{-kt}dt,$$

where

$$P = \int_0^T d(t)e^{-kt}dt \text{ and}$$

d(t) = the dividend in period t.

We note that

$$\frac{\partial \ln P}{\partial k} = \frac{1}{P}\frac{\partial P}{\partial k} = -\frac{1}{P}\int_0^T td(t)e^{-kt}dt = -D.$$

This leads to the familiar interpretation of duration as a measure of the sensitivity of asset price to discount rate:

$$\frac{\Delta P}{P} \approx -D\Delta k.$$

For a stock with constant dividend growth rate,

$$d(t) = D_0e^{gt}$$

and

$$P = \int_0^\infty D_0e^{(g-k)t}\,dt = \frac{D_0}{k-g}.$$

The duration is given by:

$$D_{DDM} = -\frac{\partial \ln P}{\partial k} = \frac{1}{k-g}.$$

Because

$$k - g = \frac{D_0}{P},$$

we recognize the duration as the reciprocal of the yield.

A helpful interpretation of the duration is obtained by computing the amount of the present value that is obtained from the first t years of the payment stream. This is given by:

$$P(t) = \int_0^t D_0e^{(g-k)t}\,dt$$

$$= \frac{D_0}{k-g}(1 - e^{(g-k)t}) = P(1 - e^{-t/D_{DDM}}).$$

It follows that:

$$P(D_{DDM}) = P\left(1 - \frac{1}{e}\right) = 0.63P.$$

That is, 63 per cent of the present value of the payment stream comes from the dividends received over a time equal to the duration. Half the present value comes from:

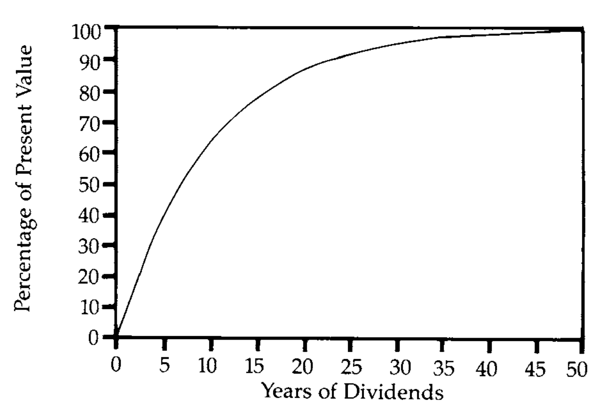
$$P(T_{1/2}) = \frac{P}{2} = P(1 - e^{-T_{1/2}/D_{DDM}}), \text{ or}$$

$$e^{-T_{1/2}/D_{DDM}} = \frac{1}{2}.$$

Taking the natural logarithm of both sides, we obtain:

$$T_{1/2} = D_{DDM}\ln 2 \approx 0.7\,D_{DDM}.$$

Figure AA Present Value of Dividends over Time



Thus, for a constant-growth stock, 50 per cent of the present value of the payment stream comes from dividends received over a time equal to approximately 70 per cent of the duration. This is depicted in Figure AA for a stock that yields 10 per cent and, therefore, has a duration equal to 10 years. As the figure illustrates, half the present value comes from the first seven years of dividends. Each successive seven-year period accounts for half the remaining present value.

Duration Using Interactive Effects

From Equation (3) in the text, with continuous compounding and cash flows, we get:

$$P = \frac{D_0}{k - g},$$

$$\ln P = \ln D_0 - \ln (k - g),$$

$$\frac{dP}{P} = d \ln P = \frac{\partial \ln P}{\partial k} dk + \frac{\partial \ln P}{\partial g} dg$$

$$= -\frac{1}{k-g}(dk - dg) = -D_{DDM}(dk - dg).$$

From Equations (5), (6) and (7) in the text we get:

$$dk = dr + dI + \frac{\partial h}{\partial I} dI + \frac{\partial h}{\partial r} dr,$$

$$dg = \gamma\, dr + \lambda\, dI.$$

We thus arrive at Equation (8):

$$\frac{dP}{P} = -D_{DDM}\left(1 - \gamma + \frac{\partial h}{\partial r}\right) dr - D_{DDM}$$

$$\left(1 - \lambda + \frac{\partial h}{\partial I}\right) dI.$$

Equation (8) provides a convenient framework for understanding the sensitivity of stock prices to changes in interest rates. The sensitivity to changes in the real rate of interest is:

$$-D_{DDM}\left(1 - \gamma + \frac{\partial h}{\partial r}\right).$$

The sensitivity to changes in interest rates resulting from changes in inflation expectations is:

$$-D_{DDM}\left(1 - \lambda + \frac{\partial h}{\partial I}\right).$$

Pension Fund Perspective footnotes concluded from page 17

7. G. Glueck, "Florida Bank Is Ordered To Sell Its Art Collection," *New York Times*, April 12, 1989, p. C-19.
8. A. Scardino, "Lush Life: Investor, 23, Named in Huge Stock Fraud," *New York Times*, January 13, 1988, p. 1.
9. A. H. Lubash, "Wall St. 'Whiz Kid' Gets 8-Year Term for Fraud," *New York Times*, December 10, 1988, p. 31.
10. C. Dickey, "A Taste for Art and Money," *Newsweek*, May 29, 1989, p. 68.
11. M. Kimmelman, "The Case of the Vanishing Art," *New York Times*, May 14, 1989, p. H-1 and H-43.