Can Noise Create the Size and Value Effects?

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Abstract

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Small-capitalization and value stocks are likely to predominantly have negative noise, while large-capitalization and growth stocks are likely to have positive noise, if prices contain random noise. Negative price noise implies that small-capitalization and value stocks are more likely undervalued and thus have higher expected return than justified by risk, while the large-capitalization and growth stocks are more likely

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overvalued. We formally verify and explore this intuition by using a standard noise-in-price model.

We compute in closed form and match quantitatively the level of and the cross-sectional variations in the expected stock return documented in Fama and French (1992). Our model is parsimonious with essentially only one adjustable parameter, the volatility of the price noise. Our study suggests that a modest amount of noise in prices can create size and value effects.

Blume and Stambaugh (1983) assume small-cap stocks have higher noise volatility and show that they have higher expected return because of Jensen's inequality. This channel is shut off in our paper because we assume all stocks have the identical return distribution thus the same noise volatility. Small-cap stocks in our paper are defined to be ones with low market capitalization and they generate higher expected returns because of the negative realization of random price noise.

1 Introduction

Noise-in-price models have been widely explored in the literature of finance and economics. In this paper, we propose a theory of the cross-section of expected stock returns based on noise in prices. We show that noise in prices can explain the size and value effects in the sense that small-cap stocks and value stocks (stocks with low price-dividend ratios or price-book ratios or price-earning ratios) have higher returns than justified by risk.

We assume that the price of a stock equals its intrinsic value, which is computed from certain economics theory or model, plus a random noise. For a given stock, we show that there is a size and value effect in time series returns; in other words, the conditional expected return decreases with the market cap and price-to-dividend ratio. This result is quite intuitive. Consider the case of a small-cap stock. The low price of such a stock may be the result of either a low intrinsic value or a negative price noise or a combination of the two reasons. A low intrinsic value just yields a fair return, but a negative noise in price implies that stock is undervalued, which leads to a superior return that is not accounted for by risk. The same logic applies to value stocks, while the reciprocal logic applies to large-capitalization and growth stocks.

We also assume that the returns of all stocks have the same distribution, that is, draws from the distribution for a single stock are the same as cross-sectional draws of different stocks. This property links time-series size and value effects to a cross-sectional size and value effects. Furthermore, in our model, because there is no cross-sectional variation in parameters, the cross-sectional variation in expected stock returns is *solely* caused by variations in the realization of noise.

We use a stylized and parsimonious noise-in-price model. We compute[1] the table of 10×10 cross-sectional expected returns that matches quantitatively to its empirical counterpart of the Fama-French (1992) table, with only one parameter that is not directly measured from

[1]We should remark that the Fama-French table is generated by simulation and matched qualitatively in other theoretical studies.

the data, namely the noise volatility of return. The other four parameters are average return, total volatility, average dividend growth rate and dividend growth volatility, none of which has any bearing on the size or value effects, and each of which can be readily estimated.

We should note that only a moderate amount of noise is needed to generate realistic size and value effects. The ratio of the variance of noise to that of total stock return is only 10%. This ratio is consistent with empirical studies on market efficiency, Poterba and Summers (1986, 1989) and Fama and French (1988).

The variation in cross-sectional returns in our model is generated by random realization of price noise, but parameter variations, such as variations in factor betas and idiosyncratic volatility will lead to additional cross-sectional variations in expected returns. Certainly, we *believe* that both parameter variation and random noise realization contribute to the cross-sectional size and value effects. However, we choose to shut off the parameter variation channel, to illustrate that noise *alone* can generate the size and value premiums documented in the literature, with a plausible noise parameter.

Berk (1995, 1997) points out that in any model in which the cross-sectional covariance between expected payoff and expected return is zero, the cross-sectional correlation between price and expected return has to be negative. Applicability of Berk's critique is wide, but it explains neither the source nor the magnitude of the size and value effects, which have been the focus of many asset-pricing studies.[2] It is important to know whether the superior return of small and value stock is a result of risk or not. Berk's critique is silent on all these issues.

Our paper proposes noise as a contributing, perhaps dominant, source of the size and value effects. We illustrate that this view is plausible by computing size and value premiums that match the empirical data. The premiums associated with small-cap and value stocks in our model are driven solely by a reasonable noise parameter and are not attributed to risk. Finally, contrary to Berk's assumption, the cross-sectional covariance between the expected

[2]To name only a few, Fama and French (1992), Lakonishok, Shleifer, and Vishny (1994), Gomez, Kogan, and Zhang (2003), Bansal and Yaron (2004), Zhang (2005), and Yogo (2006).

payoff and the expected return in our paper is nonzero and plays a crucial role in explaining the size and value effects.

Our noise-in-price model is similar to that of Blume and Stambaugh (1983). However, their mechanism that generates the size effect is completely different from ours. Blume and Stambuagh rely on a key extra assumption, that small-cap stocks have higher noise volatilities and thus have higher expected returns, because of the Jensen effect. The difference in expected stock returns in Blume and Stambaugh is largely driven by the difference in *ex ante* distribution. Furthermore, the premium from Blume and Stambaugh's mechanism is too small to explain size and value effects in lower frequency data, such as quarterly or longer. Finally, if this noise is tied to bid-ask bounce, as is the common interpretation of Blume and Stambaugh, it is inherently difficult to profitably exploit for any but the market makers. In our model, by assuming that all stocks have the same *ex ante* distributions, the Blume and Stambaugh mechanism and intuition for generating the size premium is shut off completely. Instead, small-cap stocks in our paper are defined to be ones with low market caps and, as a consequence of predominantly negative price noise, have higher expected returns. In other words, the difference in cross-sectional expected returns is completely driven by the difference in realization of the random noise shock to prices. This mechanism is not explored in Blume and Stambaugh.

Our paper is organized as follows. In Section 2, we formally introduce our model of noise. In Section 3, we solve *in closed form* the expected returns conditional on price and the price-to-dividend ratio. We then compute the table of expected returns sorted into 10×10 price and the price-to-dividend ratio deciles, which is the theoretical counterpart to Table V in Fama and French (1992). In Section 4, we discuss related literature. We make concluding remarks in Section 5.

2 Noise-in-Price Model

We present here our adaptation of the classic noise-in-price model and discuss the key economic assumptions and technical assumptions underlying the model. We note that similar specifications of the noise-in-price model are found in Blume and Stambaugh (1983), Summers (1986), Fama and French (1988), Aboody, Hughes, and Liu (2002), Arnott (2005a, b), Hsu (2006), Arnott and Hsu (2008), and Brennan and Wang (2006).

Assumption 1 *There are N stocks. At time t, stock k has an intrinsic value V_{kt}, with $k = 1, ..., N$. The intrinsic value return is a constant and is the same for all stocks.*

The observed market price, P_{kt}, of a stock deviates from its intrinsic value, V_{kt}, by noise $e^{\Delta_{kt}}$. Specifically,

$$P_{kt} = V_{kt}e^{\Delta_{kt}}, \tag{1}$$

where noise Δ_{kt} is cross-sectionally and intertemporally independent of intrinsic value V_{ls}, for stock l at time s, for all k, l, t, and s and the unconditional expectation $\mathrm{E}[e^{\Delta_{kt}}]$ of $e^{\Delta_{kt}}$ satisfies

$$\mathrm{E}[e^{\Delta_{kt}}] = 1, \quad k = 1, 2, ..., N. \tag{2}$$

The dividend, D_{kt}, of the stock is also independent of noise Δ_{ls} for stock l at time s for all k, l, t, and s.

Let $v_{kt} = \ln V_{kt}$, $p_{kt} = \ln P_{kt}$, and $d_{kt} = \ln D_{kt}$. We also assume

$$v_{kt+1} = \mu_v + v_{kt} + \beta f_{t+1} + \sigma\epsilon^v_{kt+1}, \tag{3}$$

$$\Delta_{kt+1} = -\frac{1}{2}\sigma^2_{\epsilon_\Delta} + \sigma_{\epsilon_\Delta}\epsilon^\Delta_{kt+1}, \tag{4}$$

$$v_{kt+1} - d_{kt+1} = \bar{x}_v + \sigma_{\epsilon_x}\epsilon^v_{xt+1}, \tag{5}$$

where f_t is mean-zero normal for all t that is independent of all other random variables and f_t and f_s are independent for all t and s, σ_{ϵ_Δ}, μ_v, β, σ, $\bar{x}_v$, and σ_{ϵ_x} are all constants. All shocks are independent across stocks and time.

Note that we have made our assumptions as simple as possible to illustrate that noise realization *alone* is sufficient to match the desired empirical moments. Extending the model to incorporate other realistic features is often straightforward and is discussed later.

The intrinsic value return is assumed to have a constant mean that is the same for all stocks, so we are assuming no size and value effects in the absence of noise. Equation (1) states that for each stock k, the market price P_{kt} of the stock deviates from its intrinsic value V_{kt} by a random noise $e^{\Delta_{kt}}$. Equation (2) states that the deviation, on average, is zero.

The assumptions in Equations (3-5) are mostly technical in nature and are made for tractability. Equation (3) says that the log intrinsic value is a Gaussian random walk. Equation (4) says that the log noise is normal and identically and independently distributed (*i.i.d.*) over time. Although this assumption implies that the pricing error will disappear after one period, the assumption can be relaxed without losing tractability. Equation (5) says that the price-to-dividend ratio is mean reverting over time.[3]

We assume lognormality in our random variables to gives us tractability to compute conditional expected returns. With non-gaussian specifications, it is not easy to compute in closed form the inference about the noise, but all the insights and intuitions remain. We note also that the price noise is specified in multiplicative form instead of additive form, again for tractability.[4]

In the above assumption, f_t represents the systematic risk factor, and the beta coefficient is the same for all stocks. All stocks have the same expected return and idiosyncratic volatility. The idiosyncratic shocks are independent. Stocks have identical distributions but are not independent because of the presence of systematic risk factor f_t. We deliberately restrict our model so that parameter variations are not needed to drive our results.

In the preceeding return dynamics, the theory that determines intrinsic value V_{kt} is unspecified; it can be the consumption-based asset-pricing models, the CAPM, the APT, or any other model. The exact choice will not affect the results of this paper. For convenience, we can think of V_{kt} as the discounted present value of expected future cash flows, where the

[3]The assumption on dividend D_{kt} is necessary for computing returns since dividend D_{kt+1} is part of the cashflow for $t + 1$, in addition to the price P_{kt+1}. Equation (5) is used in the literature on predictive regressions, see for example, Stambaugh (1999) and Torous and Valkanov (2005). There is no price noise in these studies; the value-dividend ratio is the price-dividend ratio.

[4]The multiplicative form of the noise specified in Assumption 1 is used in Blume and Stambaugh (1983), Fama and French (1988), and Hsu (2006). The additive form of Summers (1986) and Aboody, Hughes, and Liu (2002) is more problematic because the price noise becomes negligible over time as V_t grows. Campbell and Kyle (1993) overcome this problem by using an additive form with detrended dividends. The problem does not arise with the multiplicative form.

discount rates are determined by the return covariance with systematic risks.

3 Cross-Section of Expected Returns

We first compute the unconditional expected stock return. Then, we derive the closed-form solution for the expected return of a stock conditional on it market price and price-to-dividend ratio. Using parameters calibrated from U.S. stock market data, we then compute the table of cross-sectional average return, matched to Table V in Fama and French (1992).

3.1 Cross-Section of Unconditional Expected Returns

We first present the unconditional expected return for stocks in our noise-in-price model.

Proposition 1 (Unconditional Expected Return) *If Assumption 1 holds, then*

$$\mathrm{E}\left[\frac{P_{kt+1}+D_{kt+1}}{P_{kt}}\right]=e^{\mu+\frac{1}{2}\sigma_r^2}\left(1+e^{-\bar{x}_v+\frac{\sigma_{\epsilon_x}^2}{2}}\right)e^{\sigma_{\epsilon_\Delta}^2}. \tag{6}$$

Note that the unconditional expected intrinsic value return, which is the unconditional expected return in the absence of noise, is

$$e^{\mu+\frac{1}{2}\sigma_r^2}\left(1+e^{-\bar{x}_v+\frac{\sigma_{\epsilon_x}^2}{2}}\right).$$

Note that the unconditional expected return is independent of k; therefore, it is the same for all stocks. The cross-section of unconditional expected return in our model is trivial. The difference between the expected return and the expected intrinsic value return increases with $\sigma_{\epsilon_\Delta}^2$ and is a result of Jensen's inequality, which is driven by the variance of the random noise.

Proposition 1 is an analytic closed-form counterpart to Blume and Stambaugh's (1983) Equation (6). Blume and Stambaugh compute unconditional expected return for stocks and show that price noise resulted from bid-ask bounce increases the unconditional expected return. Additionally, the increase in expected return increases with the noise variance. They find that the higher bid-ask spread for smaller stocks explains 50 percent of the size effect documented in daily data. But the Blume and Stambaugh mechanism explains little of the value effect.

Following the same intuition from Blume and Stambaugh (1983), we could easily generate the cross-sectional variations in expected returns documented in Fama and French (1992) if we assume variations in the exogenously specified noise volatility for stocks. Although we believe that a portion of the observed variations must be driven by variations in parameters, from a modeling perspective, it is not very satisfying that the cross-sectional variation is essentially exogenously assumed. We demonstrate that the size and value effects do not require such cross-sectional variation in parameters.

In our model, we wish to examine a different aspect of the noise-in-price model. We focus purely on the effect of the random noise realization on the cross-section of stock returns. We deliberately make the extreme assumption that all stocks are *ex ante* identical in return distribution (but correlated), so the mechanism of Blume and Stambaugh (1983) is completely shut off in our model. With our model, no pattern in cross-sectional expected return variations is driven by model parameter variations.

Finally, the noise volatility in Blume and Stambaugh (1983) that is required to match the observed size premium is large. Even at a 10 percent noise volatility for small-cap stocks and 0 percent for large-cap stocks, the predicted difference in expected return would only be 1 percent, which is very small relative to the documented size premium.

3.2 Expected Stock Return Conditional on Size and Value

Before we delve into the cross-sectional results, we first compute the expected stock return conditioned on price (size) and price-to-dividend ratio (value) of a given stock. In our model, a stock with low price and low price-to-dividend ratio have higher expected return because such a stock is more likely than other stocks to be undervalued. Accordingly, this mechanism is different from that of Blume and Stambaugh (1983).

The intuition of the mechanism in our model is straightforward. If Δ_{kt} is negative, the market price of stock k is lower than its intrinsic value and the expected return with a negative Δ_{kt} is high. In reality, we do not observed the noise Δ_{kt}. However, the price P_{kt} or a price ratio provides information about Δ_{kt}. The lower the price or the price ratio, the more likely Δ_{kt} is to be negative and the stock to be undervalued.

In the Gaussian setting specified in Assumptions 1, the inference from noise conditioned on price and the price ratio can be computed in closed form.[5]

Proposition 2 (Conditional Expected Return) *Suppose Assumption 1 holds. Furthermore, assume that the prior distributions of noise Δ_{kt} and log price-to-dividend ratio x_{kt} are their unconditional distribution, that the prior distribution of log intrinsic value v_{kt} is normal with mean $\bar{v}_t$ and variance σ_{vt}^2, and that $(\Delta_{kt}, x_{kt}, v_{kt})$ are uncorrelated. Then, the expected return conditional on p_{kt} and x_{kt} is*

$$\mathrm{E}\left[\frac{P_{kt+1}+D_{kt+1}}{P_{kt}}|x_{kt}, p_{kt}\right] = \mu \frac{P_{kt}^{-\gamma_3} X_{kt}^{-\gamma_4}}{\overline{PX}_{t,-\gamma_3,-\gamma_4}}, \tag{7}$$

where $\gamma_3 = \frac{\frac{1}{\sigma_{vt}^2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}$ and $\gamma_4 = \frac{\frac{1}{\sigma_{\epsilon_x}^2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}$. $\overline{PX}_{t,\gamma_1,\gamma_2}$ is the expected value of $P_{kt}^{\gamma_1} X_{kt}^{\gamma_2}$ given that v_{kt} is a normal with mean $\bar{v}_t$ and variance σ_{vt}^2.

We assume that the correlation between log intrinsic value v_{kt} and log intrinsic value-to-dividend ratio $v_{kt} - d_{kt}$ is zero for notational simplicity. Incorporation of a nonzero correlation is straightforward.[6]

Note that the cross-sectional dependence of the conditional expectation is only through P_{kt} and X_{kt}. Thus, two stocks with the same price P_{kt} and price-dividend ratio X_{kt} have the same expected return. Naturally, if there is no noise ($\sigma_{\epsilon_\Delta} = 0$), the expected return is independent of the state variables P_{kt} and X_{kt}. In this case, the stock is fairly priced for

[5] To draw inference from noise Δ_{kt} from price p_{kt}, we need to know the prior joint distribution of v_{kt} and Δ_{kt}. It is natural to assume that the distribution of Δ_{kt} is its stationary distribution, which has a variance of $\sigma_{\epsilon_\Delta}^2$. Because v_{kt} is not stationary, there is no natural choice of distribution for v_{kt}. We assume that v_{kt} is normal with mean $\bar{v}_t$ and variance σ_{vt}^2, which can be obtained by a discounted dividend model and sensitivity analysis of it. From Assumption 1, v_{kt} and Δ_{kt} are independent.

[6] Many empirical studies analyze expected returns conditional on other price ratios, such as price-to-book ratio or price-to-earning ratio. We compute the expected returns conditional on the price-to-dividend ratio. Conceptually, the analysis applies in the same way to any price ratio dependence. We choose price-to-dividend ratio instead of other ratios to avoid additional parameters.

all levels of P_{kt} and X_{kt}; thus, conditioning on them would produce no effects in expected returns.

In our model, the size and value effects are both driven by the same source: the price noise. Conversely, both price p_{kt} and price-to-dividend ratio $p_{kt} - d_{kt}$ are noisy signals of Δ_{kt}. We assume that the correlation between v_{kt} and $v_{kt} - d_{kt}$ is zero; however, there is an imperfect correlation between p_{kt} and $p_{kt} - d_{kt}$ induced by noise Δ_{kt}. When p_{kt} is low, Δ_{kt} is likely to be negative, but we cannot be sure because intrinsic value v_{kt} is not observed. When both p_{kt} and $p_{kt} - d_{kt}$ are low, Δ_{kt} is more likely to be negative. Thus, p_{kt} and $p_{kt} - d_{kt}$ are correlated but are not substitutes for each other. Using both simultaneously provides more precise information about Δ_{kt}. In the next section, we use these time series size and value results to develop a model of cross-sectional expected returns.

3.3 Cross-Section of Expected Stock Returns

As we pointed out earlier, the cross-section of expected intrinsic value return is trivial, in the sense that all stocks have the same expected intrinsic value return. In our model, without noise, the sorted 10×10 Fama-French decile portfolios have identical expected returns, as price (size) and price-to-dividend ratio (value) are not related to the return distribution for stocks. With noise, the stocks sorted into various size and value deciles have different expected returns because price and price-to-dividend ratio provide information on the unobserved price noise. Sorting on price and price-to-dividend ratio is, in part, sorting on realized price noise. It is this aspect of sorting on noise, rather than the sorting on firm characteristics, that creates the variation of expected returns.

We next show that random noise realization alone can produce significant cross-sectional variations in expected returns with reasonable parameters. Fama and French (1992) demonstrate the size and value effects by empirically calculating a table of average returns for stocks sorted into size and value deciles. Our closed-form solution allows us to calibrate our model to data and compute the Fama-French Table. To the best of our knowledge, there are no theoretical computations that match quantitatively to the levels and patterns of return variation documented in Table.

Because all stocks have the same distribution in our model, the cross-sectional draw of different stocks is the same as different draws from the distribution of a single stock. Thus the cross-sectional average can be computed by using Proposition 2. Note that the time-series size and value effects become the cross-sectional size and value effect. The times-series size and value effects are driven by random shocks to the price of a single stock over time. The cross-sectional size and value effects are driven by random noises of the different stocks at time t.

Similar to Fama and French (1992), we first partition the (p_t, x_t) space into cells of 10×10 size, p_t, and value, x_t, deciles. Note that we have dropped the subscript k because we are now focusing on the distribution of a single stock. Note that p_t and x_t are joint normal with variances $\sqrt{\sigma_{vt}^2 + \sigma_e^2}$ and $\sqrt{\sigma_x^2 + \sigma_e^2}$ and correlation $\rho = \frac{\sigma_{\epsilon_\Delta}^2}{\sigma_{pt}\sqrt{\sigma_{\epsilon_x}^2 + \sigma_{\epsilon_\Delta}^2}}$. We first use p_{ti} to divide the p_t space into 10 deciles. For ith size decile, we further divide the x_t space into 10 deciles, using $x_{i,j} = \sqrt{\sigma_x^2 + \sigma_e^2}\delta_{i,j} + \bar{x}$, where $\delta_{i,j}$ can be solved numerically. Let $\mathrm{E}\left[f(z)|_{\underline{z}}^{\overline{z}}\right]$ denote the expectation of $f(z)$ for z between $\underline{z}$ and $\overline{z}$ for a standard normal random variable z.

Proposition 3 (Cross Section of Expected Return) *Suppose that Assumption 1 holds. Furthermore, the number of stock tends to infinity. Then, the expected return conditional on the (i, j) decile of (p_t, x_t) space is given by*

$$\mu \frac{\mathrm{E}\left[\left(N\left(p_{i+1} - \rho z\right) - N\left(p_i - \rho z\right)\right)|_{x_{i,j}}^{x_{i,j+1}}\right]}{0.01},$$

where $p_{ti} \equiv \delta_i + \left(\gamma_3\sigma_{pt} + \rho\gamma_4\sqrt{\sigma_{\epsilon_x}^2 + \sigma_{\epsilon_\Delta}^2}\right)$; $x_{i,j} \equiv \delta_{i,j} + \left(\gamma_4\sqrt{\sigma_{\epsilon_x}^2 + \sigma_{\epsilon_\Delta}^2} + \rho\gamma_3\sigma_{pt}\right)$, with $i = 1, ..., 9$; z is a standard normal random variable and the expectation is taken with respect to z. $N(\cdot)$ is the standard normal cumulative probability distribution function.

We use the first and second moments of the U.S. equity market data to calibrate the above specification. The parameters are summarized in Table 1. The parameter μ affects only the overall magnitude of the expected return; this parameter has no impact on the results of our study. We take μ to be 3 percent, which together with Jensen's effect, produces a U.S. market average return of about 8 percent. Because the mean and volatility of the price-to-dividend

Table 1: Summary of Parameters

μ	3%	expected stock return (before Jensen's effect)
σ_r	30%	volatility of intrinsic value return
σ_{ϵ_Δ}	10%	volatility of noise
$\bar{x}_v$	3.5	mean log price-dividend ratio
σ_{ϵ_x}	10%	log-dividend ratio volatility

ratio are small, the volatility of the stock return is largely caused by price fluctuations. Note that from Assumption 1,

$$p_{t+1} - p_t = v_{t+1} - v_t + e_{t+1} - e_t = \mu + \Delta_t + \sigma_r \epsilon_{rt+1} + \sigma_{\epsilon_\Delta} \epsilon_{\Delta t+1};$$

thus, the variance of the return is the sum of variance σ_r^2 of the value return $v_{t+1} - v_t$ and conditional variance $\sigma_{\epsilon_\Delta}^2$ of noise Δ_{t+1}. We take $\sigma_r = 30\%$ and $\sigma_{\epsilon_\Delta} = \sigma_r/3 \approx 10\%$. The ratio $\sigma_r/\sigma_{\epsilon_\Delta} = 3$ gives a ratio between variance of the noise and total variance of the stock return of 11 percent. French and Roll (1986) suggest that "between 4% and 12% of the daily return variances is caused by noise." Fama and French (1988) estimate that the predictable variation in stock returns, because of mean reversion, is about 35 percent of the long horizon variances, and they suggest, similar to Summers (1986), that mean reversion may be a result of market inefficiency. In his calibration exercises, Summers uses a value for σ_r^2 which is of the same order of magnitude as $\sigma_{\epsilon_\Delta}^2$.

The calibration of parameters for the intrinsic value-to-dividend ratio process is as follows. Assuming a mean dividend ratio to be about 3 percent, we take $\bar{x}_v = \ln(1/0.03) = 3.5$. We set $\sigma_{\epsilon_x} = 10\%$. The tables of expected returns are not sensitive to this parameter.[7]

We present the expected returns conditional jointly on size and value in Table 2, which is computed from Proposition 3 with the parameters given in Table 1. The intuition for the table is simple. The expected stock return computed for each 10×10 cell is the expected

[7] In Stambaugh (1999) and Torous and Valkanov (2000), the price-to-dividend ratio process for the market follows an AR(1) process. In our specification, the AR(1) coefficient is set to zero for simplicity. This parameter can be introduced in the model without losing tractability, but our empirical tables are not sensitive to this parameter.

Table 2: Expected Annual Returns Conditional on Size and Value Deciles

	Dividend-to-Price Ratio										
	All	1	2	3	4	5	6	7	8	9	10
All	11.46	5.54	7.87	9.10	10.09	10.97	11.84	12.74	13.76	15.06	17.62
Small-ME	13.78	7.90	10.22	11.44	12.42	13.30	14.16	15.05	16.06	17.35	19.89
ME-2	12.83	7.03	9.32	10.52	11.49	12.36	13.21	14.09	15.08	16.35	18.86
ME-3	12.34	6.57	8.85	10.05	11.01	11.87	12.72	13.60	14.59	15.85	18.34
ME-4	11.96	6.21	8.48	9.67	10.63	11.49	12.33	13.21	14.20	15.46	17.94
ME-5	11.62	5.88	8.14	9.34	10.29	11.15	11.99	12.86	13.85	15.10	17.57
ME-6	11.29	5.57	7.82	9.01	9.96	10.82	11.66	12.53	13.51	14.76	17.23
ME-7	10.95	5.25	7.49	8.68	9.63	10.48	11.32	12.18	13.16	14.41	16.87
ME-8	10.57	4.89	7.13	8.31	9.25	10.10	10.94	11.80	12.78	14.02	16.47
ME-9	10.09	4.43	6.66	7.84	8.78	9.63	10.46	11.32	12.29	13.53	15.97
Large-ME	9.18	3.53	5.76	6.93	7.87	8.72	9.54	10.40	11.37	12.60	15.04

This table presents annual expected returns, in percentage, conditional on price (ME) and dividend-to-price ratio deciles.

return conditional on belonging to a price and price-to-dividend ratio joint decile. Belonging to a low price and low price-to-dividend ratio decile is a signal for being more likely undervalued than overvalued.

As we pointed out earlier, we choose the price-to-dividend ratio mainly to avoid extra parameters. We would expect little or no difference if the price-to-book or price-to-earnings ratios were used instead. The expected returns in Table 2 are similar to those of Table V of Fama and French (1992), when annualized. Note that our expected returns are monotonic as a function of deciles while the monotonicity is only approximately in Table V of Fama and French; this discrepancy can easily be driven by measurement errors in the sample averages.

Berk (1995, 1997) posits that the cross-sectional correlation between the expected payoff and the expected return is zero, which implies that a negative correlation between price and expected return in the cross-section is driven entirely by the cross-sectional variance in expected returns. This position is not true in our model. To see this result, we use the following identity derived in Berk (1995):

$$\mathrm{cov}(p, r) = \mathrm{cov}(c, r) - \mathrm{var}(r),$$

where *cov* and *var* denote cross-sectional covariance and variance respectively. We can show that $\text{cov}(c, r) = -(1-\gamma_1)\gamma_1\sigma^2(p)$ and $\text{var}(r) = \gamma_1^2\sigma^2(r)$.[8] Thus, the ratio, $\frac{\text{cov}(c,r)}{-\text{var}(r)}$, is γ_1, which is the ratio of noise variance over the total variance of the prior distribution of the log intrinsic value. In our model, this ratio is 10 percent, so the negative correlation between price and expected return in the cross-section is largely driven by the negative correlation between expected payoff and expected return.

We also show that the high returns associated with small cap and value in our model are not attributable to the increased loading on the systematic risk factor, f_t, resulting from the interplay between the small-cap and value deciles and noise realization. In Table 3, we present the average betas for the size-value deciles. We assume that β in Equation (3) for all stocks is equal to 1; that is, in the absence of noise all stocks have the same exposure to the risk factor f. In the presence of noise, the beta of the price return is given by $\frac{V_t}{P_t} = e^{\Delta_t}$. The beta for a given size-value decile given in Table 3 is the average of the beta of the size-value decile. The small-cap and value stocks have slightly higher betas because of negative noise. Stocks in the smallest decile have a beta of 1.02, whereas those in the largest decile have a beta of 0.99. Similarly, stocks in the lowest dividend-to-price ratio decile have a beta of 0.98, whereas those in the highest decile have a beta of 1.03.[9]

We compute the abnormal return (alpha) in Table 4. We assume an annual risk-free return of 4 percent, which determines the market risk premium in our model. The results are largely unaffected by this assumed risk-free rate. We can then compute the risk premium and the alpha for each size-value decile by using the Table 3. Unsurprisingly, because the variation in beta is small in the cross-section, the high returns for small-cap and value stocks translate into positive alpha. Stocks in the smallest decile have an alpha of 1.67 percent, whereas those in the largest decile have an alpha of –1.18 percent. Similarly, stocks in the lowest dividend-to-price ratio decile have an alpha of -0.98 percent, wherea those in the highest dividend-to-price ratio decile have an alpha of 1.47 percent. The increase in factor

[8] If we integrate out X_t from Equation (7), the expected return conditional on P_t is given by the power $P_{tk}^{-\gamma_1}$. The next two equations follows straightforwardly.

[9] This finding is consistent Lakonishok, Shleifer, and Vishny (1994) who find that "the betas of value portfolios with respect to the value-weighted index tend to be about 0.1 higher than the betas of the glamour portfolios."

Table 3: Beta Conditional on Size and Value Deciles

	Dividend-to-Price Ratio										
	All	1	2	3	4	5	6	7	8	9	10
All	1.004	0.952	0.972	0.983	0.992	0.999	1.007	1.015	1.024	1.035	1.057
Small-ME	1.025	0.972	0.992	1.003	1.012	1.020	1.028	1.036	1.045	1.057	1.080
ME-2	1.016	0.964	0.984	0.995	1.004	1.012	1.019	1.027	1.036	1.048	1.070
ME-3	1.012	0.960	0.980	0.991	1.000	1.007	1.015	1.023	1.032	1.043	1.066
ME-4	1.008	0.956	0.977	0.988	0.996	1.004	1.012	1.019	1.028	1.040	1.062
ME-5	1.005	0.953	0.974	0.985	0.993	1.001	1.008	1.016	1.025	1.036	1.059
ME-6	1.002	0.951	0.971	0.982	0.990	0.998	1.005	1.013	1.022	1.033	1.056
ME-7	0.999	0.948	0.968	0.979	0.987	0.995	1.002	1.010	1.019	1.030	1.052
ME-8	0.996	0.944	0.965	0.975	0.984	0.991	0.999	1.007	1.016	1.027	1.049
ME-9	0.991	0.940	0.960	0.971	0.980	0.987	0.995	1.002	1.011	1.022	1.044
Large-ME	0.983	0.932	0.952	0.963	0.971	0.979	0.986	0.994	1.003	1.014	1.036

This table presents beta of price (ME) and dividend-to-price deciles. The parameters are given by Table 1.

loading driven by noise plays an insignificant role in explaining the size and value effects.

In short, our noise-in-price model would deliver a size effect, a value effect, a Lakonishok-Shleifer-Vishny (1994) beta effect, and (not explored in this paper) a long-horizon reversal effect. With a ratio of 10 percent of noise variance over total variance, we find that each of these effects (size, value, and beta) conform closely to empirical data.

We do not want readers to infer that we believe noise in price singularly causes these effects. Nor do we suggest that the intrinsic values for all stocks have identical distributions. Rather, we make these extreme simplifying assumptions to demonstrate that noise alone would suffice to *create* the empirically observed size effect, value effect and LSV beta effect, not to mention the well-documented evidence of long-horizon mean reversion. We believe that noise in price is an important, and perhaps dominant, contributor to these empirical phenomena.

4 Related Literature

The size and value effects have spurred spirited debates since Banz (1981) and Reinganum (1981) documented that small-cap stocks tend to outperform on a risk-adjusted basis and

Table 4: Alpha Conditional on Size and Value Deciles

	Dividend-to-Price Ratio										
	All	1	2	3	4	5	6	7	8	9	10
All	0.38	-4.99	-2.88	-1.76	-0.87	-0.06	0.72	1.54	2.46	3.64	5.95
Small-ME	2.55	-2.96	-0.79	0.36	1.27	2.10	2.90	3.74	4.69	5.89	8.27
ME-2	1.66	-3.77	-1.63	-0.50	0.40	1.22	2.01	2.84	3.77	4.96	7.30
ME-3	1.20	-4.20	-2.07	-0.95	-0.05	0.76	1.55	2.38	3.30	4.49	6.82
ME-4	0.84	-4.54	-2.42	-1.30	-0.40	0.40	1.19	2.01	2.94	4.12	6.44
ME-5	0.52	-4.85	-2.73	-1.61	-0.72	0.08	0.87	1.69	2.61	3.79	6.10
ME-6	0.21	-5.14	-3.03	-1.92	-1.03	-0.22	0.56	1.38	2.29	3.47	5.78
ME-7	-0.11	-5.44	-3.34	-2.23	-1.34	-0.54	0.24	1.05	1.97	3.14	5.44
ME-8	-0.46	-5.78	-3.68	-2.58	-1.69	-0.90	-0.12	0.69	1.61	2.77	5.07
ME-9	-0.91	-6.21	-4.12	-3.02	-2.13	-1.34	-0.56	0.24	1.15	2.31	4.60
Large-ME	-1.76	-7.05	-4.96	-3.87	-2.99	-2.20	-1.42	-0.62	0.29	1.44	3.72

This table presents annual alpha, in percentage, of price (ME) and dividend-to-price deciles. The parameters are given by Table 1 and the riskfree return is assumed to be 4%.

Stattman (1980) and Rosenberg and Reid and Lanstein (1985) documented that high book-to-market ratio stocks also outperform. Similarly, other ratios such as earning-to-price (documented by Basu (1977)) and dividend yield (documented by Razeff (1984), Shiller (1984), Blume (1980) and Keim (1985)) also predict future performance.

Berk (1995, 1997) points out that cross-sectional dispersion in expected return leads to negative cross-sectional correlation between price and expected returns, in most reasonable models, regardless whether rational or behavioral. Thus, qualitatively predicting size and value effects is not a distinguishing model feature. The hard work, then, lies in identifying the mechanism of the size and value effects, matching the magnitudes to the observed levels and variations in the cross-section of stock returns, with reasonable parameters, and generating additional intuitions and testable implications.

Fama and French (1992) show that size and value, together with market beta, capture much of the cross-sectional variation in stock returns and subsume the explanatory powers of other financial variables. They propose that the size and value premia are compensation for risk. Gomes, Kogan and Zhang (2003) and Zhang (2005) explore the value effects through irreversible investments. Bansal and Yaron (2004) argue that long run risk can be used to explain cross-sectional patterns of stock return. Yogo (2006) proposes that the size and value effects can be explained by investor preferences that are non-separable in nondurable and durable consumption.

Fama and French (2007) study the pattern of decile migration amongst small-cap and large-cap stocks and value and growth stocks, and demonstrate that this migration is an important contributor to the size and value premia. Chen and Zhao (2008) reaches similar conclusions. Our model predicts that mean reversion in noise leads to decile migration and size and value premium. It remains to compute the premium associated with various migration patterns in our model and compare them with the Fama-French findings.

Blume and Stambaugh (1983) point out that the unconditional expected return of a stock in the presence of price noise increases with the noise volatility because of Jensen's inequality effect. Small-cap stocks probably have a higher noise volatility, as empirically evidence would support, which leads to a higher expected return. Even though the Blume and Stambaugh study is motivated by bid-ask random bounce, their results are applicable to general noise-in-price models.

We would argue that the average return for size-value decile portfolios corresponds to expected return conditional on size and value decile. Furthermore, one probably needs to specify noise volatilities exogenously, for 100 decile portfolios, which implies a proliferation of parameters. It is not clear how to assign these parameters other than backing them out from the Fama-French tables. Finally, these noise volatilities implies unreasonable amount of price noise. In our paper, the mechanism of Blume and Stambaugh is completely shut off by our assuming that all stocks have the same noise volatility, thus the same unconditional expected returns. Indeed, if we introduce a crosssectional variation in noise volatility with size, then an even smaller noise volatility can fully explain the observed size and value effects.

The behavioral finance literature provides additional structure and intuition for the noise-in-price model. Noise or pricing error can arise from investor overreaction or underreaction, as suggested by Shiller (1981), DeBondt and Thaler (1985, 1987), and Lakonishok, Shleifer and Vishny (1994), among others. Additionally, we note that in term-structure models, where the number of shocks is smaller than the number of independent securities, the general assumption is that the market prices for bonds are different from their fair values by a noise.

Although the noise-in-price framework we use is simple and stylized, it is not narrow, nor are the model implications obvious. The framework is, in fact, surprisingly rich in its applications. Specifically, Blume and Stambaugh (1983) use a noise-in-price model to study the effect of bid-ask bounce on expected returns. Campbell and Kyle (1993) use the framework to explain the high volatility and predictability of U.S. stock returns. Hughes, Liu, and Liu (2006) use the framework to explain why less transparent firms would have a higher cost of capital after controlling for risk. Arnott (2005a, 2005b) suggests this model as a probable key driver in the value and size effects, which this article formalizes. Hsu (2006) uses the framework to argue that a mispricing premium may exist because there are investors with liquidity needs. Hsu (2008) and Arnott and Hsu (2008) use the noise model to demonstrate that a diversified capitalization-weighted portfolio is suboptimal to any diversified non-price-weighted portfolio. Brennan and Wang (2006), using a similar framework, study the effect of mispricing on unconditional expected returns for a larger class of pricing models.

In our model, abnormal returns can be earned by exploiting size and value as signals for undervaluation. They are arguably two sides of the same coin. Summers (1986) argues that "data in conjunction with current [econometric] methods provide no evidence against the view that financial market prices deviate widely and frequently from rational valuations." Black (1986) argues that noise should always be present because investors are risk averse and are not sure whether a free lunch is truly a free lunch. According to Black, "noise creates the opportunities to trade profitably, but at the same time makes it difficult to trade profitably." If Black is right, size and value effects are likely to continue to persist. Fama and French (1988) and Poterba and Summers (1988) study mean reversion in prices and posit that one possible explanation is mean-reverting price noise.

5 Conclusion

We use a classic noise-in-price model to produce new insight into the role of price noise as a source for cross-sectional variations in expected returns. Even with no variation in unconditional expected returns, small-cap and value stocks have higher expected returns because they are more likely to be undervalued as a result of negative price shocks. With only one parameter that is not measured directly from the data (this parameter being noise volatility, which is 10 percent per year), we calculate in closed form a table of expected returns conditional on size and value deciles, which match quantitatively the table of empirical cross-sectional returns reported in Fama and French (1992). Our results suggest that a modest amount of noise can explain the entire size and value premia.

We emphasize that the variation in expected returns is completely due to variation in noise realization, not variation in noise volatility. This feature not only makes the model parsimonious but also is a key to producing large size and value premium. To the best of our knowledge, this is the first paper that theoretically computes[10] and matches quantitatively Table V of Fama-French (1992). This is possible, in part, because we have a simple mechanism and tractable model with only five parameters.

In this paper, we assumed that the *ex ante* distributions for all stocks are identical and used noise realization to generate the size and value premia in the cross-section. We deliberately make this unrealistic assumption to demonstrate that the random realization of noise alone can generate sufficient cross-sectional variation to match the data. Introducing differences in *ex ante* distributions would introduce more variations in the cross-section of expected returns and would allow us to match the emprical evidence with even less noise volatility.

We can extend our model in some directions without losing tractability. For example, we can allow the noise process to be an AR(1) process instead of *i.i.d.* Empirical studies

[10]In other studies, the Fama-French table is generated by simulate and matched qualitatively, presumably because of model complexity.

document that the size and value premiums in booms are different from those in recessions. In our paper, we can introduce a dependence of the conditional variance of noise on macroeconomic state variables. This condition can then generate a business cycle pattern in the size and value effects.

In short, a moderate amount of noise in price, which is perhaps too small to discern statistically, conforming to Summers (1986), can help explain empirical regularities like the size and value premia in stock returns and also offer extensive opportunities for further research.

Appendix

The following lemma is a special case studied in Liptser and Shiryaev (1977).

Lemma 1 *Suppose that θ is a vector of normal random variables with mean vector $\bar{\theta}$ and variance-covariance matrix Σ_θ. Furthermore, a vector of random variables ξ satisfies*

$$\xi = A_0 + A_1\theta + B\epsilon,$$

where ϵ is a vector of standard normal random variables that are independent of θ. Assume that $A_1\Sigma_\theta A_1' + BB'$ is invertible. Then, mean vector $\mathrm{E}[\theta|\xi]$ of θ conditional on ξ and the variance-covariance matrix $\Sigma_{\theta|\xi}$ conditional on ξ are

$$\mathrm{E}[\theta|\xi] = \bar{\theta} + \Sigma_\theta A_1'(A_1\Sigma_\theta A_1' + BB')^{-1}(\xi - A_0 - A_1\bar{\theta})$$

and

$$\Sigma_{\theta|\xi} = \Sigma_\theta - \Sigma_\theta A_1'(A_1\Sigma_\theta A_1' + BB')^{-1}A_1\Sigma_\theta.$$

We will apply this lemma repeatedly. In our applications, θ will be the noise Δ_t, ξ will be the price p_t or the price-to-dividend ratio $p_t - d_t$, and ϵ will be the other random variables, such as ϵ_{rt} (or f_t later).

Proof of Proposition 1

Under the assumption of Proposition 1, we have

$$\begin{aligned} \mathrm{E}[e^{\Delta_{kt+1}-\Delta_{kt}}] &= \mathrm{E}[e^{\Delta_{kt}}]\mathrm{E}[e^{\sigma_{\epsilon_\Delta}\epsilon_{kt+1}}] = e^{\frac{\sigma^2_{\epsilon_\Delta}}{2}}e^{\frac{\sigma^2_{\epsilon_\Delta}}{2}} = e^{\sigma^2_{\epsilon_\Delta}}; \\ \mathrm{E}[e^{\Delta_{kt}}]\mathrm{E}\left[\frac{1}{e^{\Delta_{kt}}}\right] &= e^{\frac{\sigma^2_{\epsilon_\Delta}}{1-\rho^2}}, \end{aligned}$$

where $\sigma^2_{\epsilon_\Delta}$ is the unconditional variance of Δ_{kt}. Because $e^{\sigma^2_{\epsilon_\Delta}} \geq 1$ and $e^{\sigma^2_{\epsilon_\Delta}} \geq 1$, we conclude that $\mathrm{E}\left[\frac{P_{kt+1}+D_{kt+1}}{P_{kt}}\right] \geq \mathrm{E}\left[\frac{V_{kt+1}+D_{kt+1}}{V_{kt}}\right]$. Equation (6) is proved by noting that

$$\begin{aligned} \mathrm{E}\left[\frac{V_{kt+1}}{V_{kt}}\right] &= e^{\mu+\frac{1}{2}\sigma_r^2}; \\ \mathrm{E}\left[\frac{D_{kt+1}}{V_{kt}}\right] &= \mathrm{E}\left[\frac{V_{kt+1}}{V_{kt}}\right]\mathrm{E}\left[\frac{D_{kt+1}}{V_{kt+1}}\right] = e^{\mu+\frac{1}{2}\sigma_r^2}e^{-\bar{x}_v+\frac{\sigma^2_{\epsilon_x}}{2}}. \end{aligned}$$

Proof of Proposition 2

At time t, we have two signals on Δ_{kt}:

$$\begin{aligned} p_{kt} &= v_{kt} + \Delta_{kt} - \ln(\mathrm{E}[e^{\Delta_{kt}}]); \\ x_{kt} &= (v_{kt} - d_{kt}) + \Delta_{kt} - \ln(\mathrm{E}[e^{\Delta_{kt}}]). \end{aligned}$$

Note that v_{kt}, $v_{kt} - d_{kt}$, and Δ_{kt} have a normal distribution with mean $(\bar{v}_t, \bar{x}_v, 0)$ and a diagonal covariance matrix with diagonal covariance matrix elements of $\left(\sigma_{vt}^2, \sigma_{\epsilon_x}^2, \sigma_{\epsilon_\Delta}^2\right)$. We can express the above equation as

$$\begin{aligned} p_{kt} - \bar{v}_{kt} + \ln(\mathrm{E}[e^{\Delta_{kt}}]) &= (v_{kt} - \bar{v}_{kt}) + \Delta_{kt}; \\ x_{kt} - \bar{x} &= (v_{kt} - d_{kt} - \bar{x}_v) + \Delta_{kt}. \end{aligned}$$

Therefore, conditional on p_{kt} and x_{kt}, the mean of Δ_{kt} is

$$\frac{\frac{1}{\sigma_{vt}^2}(p_{kt} - \bar{p}_{kt}) + \frac{1}{\sigma_{\epsilon_x}^2}(x_{kt} - \bar{x})}{\frac{1}{\sigma_{vt}^2} + \frac{1}{\sigma_{\epsilon_x}^2} + \frac{1}{\sigma_{\epsilon_\Delta}^2}}$$

and the variance is

$$\frac{1}{\frac{1}{\sigma_{vt}^2} + \frac{1}{\sigma_{\epsilon_x}^2} + \frac{1}{\sigma_{\epsilon_\Delta}^2}}.$$

Thus,

$$\begin{aligned} \mathrm{E}\left[e^{-\Delta_{kt}} | p_{kt}, x_{kt}\right] &= e^{-\frac{\frac{1}{\sigma_{vt}^2}(p_{kt} - \bar{p}_{kt}) + \frac{1}{\sigma_{\epsilon_x}^2}(x_{kt} - \bar{x})}{\frac{1}{\sigma_{vt}^2} + \frac{1}{\sigma_{\epsilon_x}^2} + \frac{1}{\sigma_{\epsilon_\Delta}^2}}} e^{\frac{1}{2\left(\frac{1}{\sigma_{vt}^2} + \frac{1}{\sigma_{\epsilon_x}^2} + \frac{1}{\sigma_{\epsilon_\Delta}^2}\right)}} \\ &= e^{\frac{\left(\frac{1}{\sigma_{vt}^2}\bar{p}_{kt} + \frac{1}{\sigma_{\epsilon_x}^2}\bar{x}\right) + \frac{1}{2}}{\frac{1}{\sigma_{vt}^2} + \frac{1}{\sigma_{\epsilon_x}^2} + \frac{1}{\sigma_{\epsilon_\Delta}^2}}} e^{\frac{\frac{-1}{\sigma_{vt}^2} p_{kt}}{\frac{1}{\sigma_{vt}^2} + \frac{1}{\sigma_{\epsilon_x}^2} + \frac{1}{\sigma_{\epsilon_\Delta}^2}}} e^{\frac{-\frac{1}{\sigma_{\epsilon_x}^2} x_{kt}}{\frac{1}{\sigma_{vt}^2} + \frac{1}{\sigma_{\epsilon_x}^2} + \frac{1}{\sigma_{\epsilon_\Delta}^2}}}. \end{aligned}$$

The first equality of the equation in the proposition is obtained by noting that

$$\mathrm{E}\left[\frac{P_{kt+1}}{P_{kt}} | x_{kt}, p_{kt}\right] = e^{\mu + \frac{1}{2}(\sigma_r^2 + \sigma_{\epsilon_\Delta}^2)} \mathrm{E}\left[e^{-\Delta_{kt}} | x_{kt}, p_{kt}\right].$$

The second equality follows from definitions $P_{kt} = e^{p_{kt}}$ and $X_{kt} = e^{x_{kt}}$. Note that

$$v_{kt+1} - d_{kt+1} = \bar{x}_v + \sigma_{\epsilon_x} \epsilon_{kt+1}^x;$$

$$
\begin{aligned}
\mathrm{E}\left[\frac{D_{kt+1}}{P_{kt}}|x_{kt}\right] &= \mathrm{E}\left[\frac{D_{kt+1}}{V_{kt}e^{\Delta_{kt}-\ln(\mathrm{E}[e^{\Delta_{kt}}])}}|x_{kt}\right] = \mathrm{E}\left[\frac{V_{kt+1}}{V_{kt}}\frac{D_{kt+1}}{V_{kt+1}}e^{-\Delta_{kt}+\ln(\mathrm{E}[e^{\Delta_{kt}}])}|p_{kt}, x_{kt}\right] \\
&= e^{\mu+\frac{1}{2}(\sigma_r^2+\sigma_{\epsilon_x}^2)-\bar{x}}\mathrm{E}\left[e^{-\Delta_t}|p_{kt}, x_{kt}\right].
\end{aligned}
$$

Finally,

$$
\begin{aligned}
& \mathrm{E}\left[\frac{P_{kt+1}+D_{kt+1}}{P_{kt}}|x_{kt}, p_{kt}\right] = \mathrm{E}\left[\frac{P_{kt+1}}{P_{kt}}|x_{kt}\right] + \mathrm{E}\left[\frac{D_{kt+1}}{V_{kt}e^{\Delta_{kt}}}|x_{kt}, p_{kt}\right] \\
= \; & e^{\mu+\frac{1}{2}(\sigma_r^2+\sigma_{\epsilon_\Delta}^2)}\mathrm{E}\left[e^{-\Delta_{kt}}|x_{kt}, p_{kt}\right] + e^{\mu+\frac{1}{2}(\sigma_r^2+\sigma_{\epsilon_x}^2)-\bar{x}}\mathrm{E}\left[e^{-\Delta_{kt}}|x_{kt}, p_{kt}\right] \\
= \; & e^{\mu+\frac{1}{2}(\sigma_r^2+\sigma_{\epsilon_\Delta}^2)} e^{\frac{\frac{1}{2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} e^{\frac{\frac{-1}{\sigma_{vt}^2}(p_{kt}-\bar{p}_{kt})}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} e^{\frac{\frac{-1}{\sigma_{\epsilon_x}^2}(x_{kt}-\bar{x})}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} \\
& + e^{\mu+\frac{1}{2}(\sigma_r^2+\sigma_{\epsilon_x}^2)-\bar{x}} e^{\frac{\frac{1}{2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} e^{\frac{\frac{-1}{\sigma_{vt}^2}(p_{kt}-\bar{p}_{kt})}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} e^{\frac{\frac{-1}{\sigma_{\epsilon_x}^2}(x_{kt}-\bar{x})}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} \\
= \; & e^{\mu+\frac{1}{2}(\sigma_r^2+\sigma_{\epsilon_\Delta}^2)} e^{\frac{\left(\frac{1}{\sigma_{vt}^2}\bar{p}_t+\frac{1}{\sigma_{\epsilon_x}^2}\bar{x}\right)+\frac{1}{2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} P_{kt}^{\frac{\frac{-1}{\sigma_{vt}^2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} X_{kt}^{\frac{\frac{-1}{\sigma_{\epsilon_x}^2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} \\
& + e^{\mu+\frac{1}{2}(\sigma_r^2+\sigma_{\epsilon_x}^2)-\bar{x}} e^{\frac{\left(\frac{1}{\sigma_{vt}^2}\bar{p}_{kt}+\frac{1}{\sigma_{\epsilon_x}^2}\bar{x}\right)+\frac{1}{2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} P_{kt}^{\frac{\frac{-1}{\sigma_{vt}^2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}} X_{kt}^{\frac{\frac{-1}{\sigma_{\epsilon_x}^2}}{\frac{1}{\sigma_{vt}^2}+\frac{1}{\sigma_{\epsilon_x}^2}+\frac{1}{\sigma_{\epsilon_\Delta}^2}}}.
\end{aligned}
$$

It is straightforward to evaluate the expectations in the proposition and prove the equivalence between the above equation and the equation given in the proposition.

Proof of Proposition 3

We will denote $p_t = (p_{1t}, ..., p_{Nt})'$. In vector notation, we can write

$$p_t = \bar{p}_t + \beta f_t + \sigma\epsilon_t^v + \sigma_{\epsilon_\Delta}\epsilon_t^\Delta.$$

We can write

$$p_t - \bar{p}_t = \sigma_{\epsilon_\Delta}\epsilon_t^v + \beta f_t + \sigma\epsilon_t^v.$$

In terms of the notation of Lemma 1, $\theta = \sigma_{\epsilon_\Delta}\epsilon_t^e$, $\bar{\theta} = 0$, $A_0 = 0$, $A_1 = I$ (where I is the N-dimensional identity matrix), $B = (\sigma I, \beta I_1)$, where I_1 is a $N \times 1$ vector of 1's. Therefore,

$$\Sigma_\theta = \sigma^2 I$$

and

$$A_1\Sigma_\theta A_1' + BB' = (\sigma^2 + \sigma^2_{\epsilon_\Delta})I + \beta^2 I_1 I_1'.$$

Let $D = \sigma^2 + \sigma^2_{\epsilon_\Delta}$ and $\beta_0 = \beta + \beta_e$, we get

$$(A_1\Sigma_\theta A_1' + BB')^{-1} = (\sigma^2 + \sigma^2_{\epsilon_\Delta})^{-1} I - (\sigma^2 + \sigma^2_{\epsilon_\Delta})^{-2}\beta^2(1 + N\beta^2(\sigma^2 + \sigma^2_{\epsilon_\Delta})^{-1})I_1 I_1'.$$

An application of Lemma 1 implies that

$$\begin{aligned}
\bar{\Delta}_t &= \Sigma_\theta A_1'(A_1\Sigma_{\theta_0} A_1' + BB')^{-1}\xi \\
&= \sigma^2_{\epsilon_\Delta}(\sigma^2 + \sigma^2_{\epsilon_\Delta})^{-1}(p_t - \bar{p}_t) - \sigma^2_{\epsilon_\Delta}(\sigma^2 + \sigma^2_{\epsilon_\Delta})^{-2}\beta^2(1 + N\beta^2(\sigma^2 + \sigma^2_{\epsilon_\Delta})^{-1})^{-1}(p_t - \bar{p}_t).
\end{aligned}$$

The first term corresponds to the case of $\beta = 0$.

When $N \to \infty$, $(1 + N\beta^2(\sigma^2 + \sigma^2_{\epsilon_\Delta})^{-1})^{-1} \to 0$; thus the second term goes to zero. The above formula reduces to the formula for the case[11] of $\beta = 0$:

$$\bar{\Delta}_t = \sigma^2_{\epsilon_\Delta}(\sigma^2_{\epsilon_\Delta} + \sigma^2)^{-1}(p_t - \bar{p}_t).$$

Intuitively, each stock price is a signal on f_t. With infinitely many stocks thus infinitely many signals, the factor uncertainty is eliminated and can be ignored for inferences about noise Δ_t and in the computation of the expected return conditional on prices and price ratios. We need to consider only the case of a single stock as long as only idiosyncratic volatility σ is used. Thus in the remainder of the proof, we will drop the subscript k for individual stock.

Now let $\sigma_{xt} = \sqrt{\frac{1}{\sigma^2_{\epsilon_x}} + \frac{1}{\sigma^2_{\epsilon_\Delta}}}$. Without loss of generality, we can assume that the means of p_t and x_t are zero. We need to compute

$$\mathrm{E}[e^{-(\phi_1 p_t + \phi_2 x_t)}|R_1],$$

where $R_1 = \{\sigma_{pt}\delta_i \leq p_t \leq \sigma_{pt}\delta_{i+1}, \sigma_{xt}\delta_{i,j} \leq x_t \leq \sigma_{xt}\delta_{i,j+1}\}$, for various ϕ_1 and ϕ_2. Define q and z by the following equations:

$$\begin{aligned}
p_t &= \sqrt{1-\rho^2}\sigma_{pt} q + \rho\sigma_{pt} z, \\
x_t &= \sigma_{xt} z.
\end{aligned}$$

[11]Note that in Proposition 2, the variance of Δ_{kt} is $\sigma^2_{\epsilon_\Delta}$ and variance of the value process v_{kt} is σ^2_{vt}.

Using the fact that p_t and x_t have variances of σ^2_{pt} and σ^2_{xt} and covariance of $\rho\sigma_{pt}\sigma_{xt}$, we can show that q and z are independent standard normals. By changing the variable from (p_t, x_t) to (q, z), we get,

$$\begin{aligned} & \mathrm{E}[e^{-(\phi_1 p_t+\phi_2 x_t)}|R_1] = \mathrm{E}[e^{-(\phi_1(\sqrt{1-\rho^2}\sigma_{pt}q+\rho\sigma_{pt}z)+\phi_2\sigma_{xt}z)}|R_2] \\ = \ & \mathrm{E}[e^{-\phi_1\sqrt{1-\rho^2}\sigma_{pt}q-(\phi_1\rho\sigma_{pt}+\phi_2\sigma_{xt})z}|R_2], \end{aligned}$$

where $R_2 = \{\delta_i \le \sqrt{1-\rho^2}q + \rho z \le \delta_{i+1}, \delta_{i,j} \le z \le \delta_{i,j+1}\}$. Integrating out q, we get,

$$\mathrm{E}[e^{-\phi_1\sqrt{1-\rho^2}\sigma_{pt}q-(\phi_1\rho\sigma_{pt}+\phi_2\sigma_{xt})z}|R_2] = e^{\frac{1}{2}\phi_1^2\sigma_{pt}^2(1-\rho^2)}\mathrm{E}[e^{-(\phi_1\rho\sigma_{pt}+\phi_2\sigma_{xt})z}(N(x_1)-N(x_2))|R_3],$$

where $x_1 = \frac{\delta_{i+1}-\rho z}{\sqrt{1-\rho^2}} + \phi_1\sqrt{1-\rho^2}\sigma_{pt}$, $x_2 = \frac{\delta_i-\rho z}{\sqrt{1-\rho^2}} + \phi_1\sqrt{1-\rho^2}\sigma_{pt}$, and $R_3 = (\delta_{i,j}, \delta_{i,j+1})$. We can show that

$$\begin{aligned} & e^{\frac{1}{2}\phi_1^2\sigma_{pt}^2(1-\rho^2)}\mathrm{E}[e^{-(\phi_1\rho\sigma_{pt}+\phi_2\sigma_{xt})z}(N(x_1)-N(x_2))|R_3] \\ & = e^{\frac{1}{2}(\phi_1^2\sigma_{pt}^2+\rho\phi_1\phi_2\sigma_{pt}\sigma_{xt}+\phi_2^2\sigma_{xt}^2)}\mathrm{E}[(N(x_3)-N(x_3))|R_4] \\ & = \mathrm{E}[e^{-(\phi_1 p_t+\phi_2 x_t)}(N(x_3)-N(x_4))|R_4], \end{aligned}$$

where $x_3 = \frac{\delta_{i+1}-\rho z+(\phi_2\rho\sigma_{xt}+\phi_1\sigma_{pt})}{\sqrt{1-\rho^2}}$, $x_4 = \frac{\delta_i-\rho z+(\phi_2\rho\sigma_{xt}+\phi_1\sigma_{pt})}{\sqrt{1-\rho^2}}$, and $R_4 = (\delta_j+\phi_1\rho\sigma_{pt}+\phi_2\sigma_{xt}, \delta_{j+1}+\phi_1\rho\sigma_{pt}+\phi_2\sigma_{xt})$. Noting that $e^{\frac{1}{2}(\phi_1^2\sigma_{pt}^2+\rho\phi_1\phi_2\sigma_{pt}\sigma_{xt}+\phi_2^2\sigma_{xt}^2)}\mathrm{E}[= \mathrm{E}[e^{-(\phi_1 p_t+\phi_2 x_t)}]$, we get

$$\mathrm{E}[e^{-(\phi_1 p_t+\phi_2 x_t)}|R_1] = \mathrm{E}[e^{-(\phi_1 p_t+\phi_2 x_t)}]\mathrm{E}[(N(x_3)-N(x_4))|R_4].$$

The proposition is proved by noting that the expected value of $e^{-(\phi_1 p_t+\phi_2 x_t)}$ conditional on R_1 is $\mathrm{E}[e^{-(\phi_1 p_t+\phi_2 x_t)}|R_1]$ divided by the probability of R_1, which is 0.01.

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